

费定晖 周学圣编演
郭大钧 邵品琮主审

Б. П. 吉米多维奇
Б. П. ДЕМИДОВИЧ

数学分析 习题集题解

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第三章 不定积分

§ 1. 最简单的不定积分

1° 不定积分的概念 若 $f(x)$ 为连续函数及 $F'(x) = f(x)$, 则

$$\int f(x) dx = F(x) + C,$$

式中 C 为任意常数.

2° 不定积分的基本性质:

$$(a) \ d\left[\int f(x) dx\right] = f(x) dx; (b) \int d\Phi(x) = \Phi(x) + C;$$

$$(b) \int Af(x) dx = A \int f(x) dx \quad (A = \text{常数});$$

$$(r) \int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx.$$

3° 最简积分表:

$$I. \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1);$$

$$II. \int \frac{dx}{x} = \ln|x| + C (x \neq 0);$$

$$III. \int \frac{dx}{1+x^2} = \begin{cases} \operatorname{arctg} x + C, \\ -\operatorname{arctg} x + C; \end{cases}$$

$$IV. \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C;$$

$$V. \int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C; \end{cases}$$

$$VI. \int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln|x + \sqrt{x^2 \pm 1}| + C;$$

$$\text{VI. } \int a^x dx = \frac{a^x}{\ln a} + C (a > 0, a \neq 1); \int e^x dx = e^x + C;$$

$$\text{VII. } \int \sin x dx = -\cos x + C;$$

$$\text{IX. } \int \cos x dx = \sin x + C;$$

$$\text{X. } \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C;$$

$$\text{XI. } \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C;$$

$$\text{XII. } \int \operatorname{sh} x dx = \operatorname{ch} x + C;$$

$$\text{XIII. } \int \operatorname{ch} x dx = \operatorname{sh} x + C;$$

$$\text{XIV. } \int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C;$$

$$\text{XV. } \int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C.$$

4° 积分的基本方法

(a) 引入新变数法 若

$$\int f(x) dx = F(x) + C,$$

则 $\int f(u) du = F(u) + C$, 式中 $u = \varphi(x)$.

(6) 分项积分法 若

$$f(x) = f_1(x) + f_2(x),$$

则 $\int f(x) dx = \int f_1(x) dx + \int f_2(x) dx.$

(B) 代入法 假设

$x = \varphi(t)$, 式中 $\varphi(t)$ 及其导函数 $\varphi'(t)$ 为连续的,

则得 $\int f(x) dx = \int f[\varphi(t)] \varphi'(t) dt.$

(r) 分部积分法 若 u 和 v 为 x 的可微分函数,

则 $\int u dv = uv - \int v du.$

利用最简积分表, 求出下列积分*:

$$1628. \int (3 - x^2)^3 dx.$$

$$\begin{aligned} \text{解} \quad \int (3 - x^2)^3 dx &= \int (27 - 27x^2 + 9x^4 - x^6) dx \\ &= 27x - 9x^3 + \frac{9}{5}x^5 - \frac{1}{7}x^7 + C. \end{aligned}$$

$$1629. \int x^2(5 - x)^4 dx.$$

$$\begin{aligned} \text{解} \quad \int x^2(5 - x)^4 dx &= \int (625x^2 - 500x^3 + 150x^4 - 20x^5 + x^6) dx \\ &= \frac{625}{3}x^3 - 125x^4 + 30x^5 - \frac{10}{3}x^6 + \frac{1}{7}x^7 + C. \end{aligned}$$

$$1630. \int (1 - x)(1 - 2x)(1 - 3x) dx.$$

$$\begin{aligned} \text{解} \quad \int (1 - x)(1 - 2x)(1 - 3x) dx &= \int (1 - 6x + 11x^2 - 6x^3) dx \\ &= x - 3x^2 + \frac{11}{3}x^3 - \frac{3}{2}x^4 + C. \end{aligned}$$

$$1631. \int \left(\frac{1 - x}{x} \right)^2 dx.$$

$$\text{解} \quad \int \left(\frac{1 - x}{x} \right)^2 dx = \int \left(\frac{1}{x^2} - \frac{2}{x} + 1 \right) dx$$

* 本章在叙述习题及其解答过程中, 凡出现的函数, 无论是被积函数还是原函数, 均默认是在有意义的定义域上进行的. 例如最简积分表中 I 里当 $n \leq -2$ 时, 要求 $x \neq 0$; IV 中要求 $|x| \neq 1$; V 中要求 $|x| < 1$; 以及 VI 中, 当取负号时要求 $|x| > 1$; 等等, 就未加声明. 在题解中也有相当多的类似情况. 因此, 如无特别声明, 在一般情形下, 这些定义域是很容易被读者确定的, 此处就不再予以一一指明.

$$= -\frac{1}{x} - 2\ln|x| + x + C.$$

$$1632. \int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3} \right) dx.$$

$$\text{解} \quad \int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3} \right) dx = a\ln|x| - \frac{a^2}{x} - \frac{a^3}{2x^2} + C.$$

$$1633. \int \frac{x+1}{\sqrt{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x+1}{\sqrt{x}} dx &= \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx \\ &= \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + C. \end{aligned}$$

$$1634. \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx &= \int (x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}}) dx \\ &= \frac{4}{5}x\sqrt[4]{x} - \frac{24}{17}x\sqrt[12]{x^5} + \frac{4}{3}\sqrt[4]{x^3} + C. \end{aligned}$$

$$1635. \int \frac{(1-x)^3}{x\sqrt[3]{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{(1-x)^3}{x\sqrt[3]{x}} dx &= \int (x^{-\frac{4}{3}} - 3x^{-\frac{1}{3}} + 3x^{\frac{2}{3}} - x^{\frac{5}{3}}) dx \\ &= -\frac{3}{\sqrt[3]{x}} \left(1 + \frac{3}{2}x - \frac{3}{5}x^2 + \frac{1}{8}x^3 \right) + C. \end{aligned}$$

$$1636. \int \left(1 - \frac{1}{x^2} \right) \sqrt{x} \sqrt{x} dx.$$

$$\begin{aligned}\text{解} \quad \int \left(1 - \frac{1}{x^2} \right) \sqrt{x} \sqrt{x} dx &= \int (x^{\frac{3}{4}} - x^{-\frac{5}{4}}) dx \\ &= \frac{4}{7} x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C = \frac{4(x^2 + 7)}{7 \sqrt[4]{x}} + C.\end{aligned}$$

$$1637. \int \frac{(\sqrt{2x} - \sqrt[3]{3x})^2}{x} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{(\sqrt{2x} - \sqrt[3]{3x})^2}{x} dx \\ &= \int (2 - 2 \sqrt[6]{72} x^{-\frac{1}{6}} + \sqrt[3]{9} x^{-\frac{1}{3}}) dx \\ &= 2x - \frac{12}{5} \sqrt[6]{72x^5} + \frac{3}{2} \sqrt[3]{9x^2} + C.\end{aligned}$$

$$1638. \int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx \\ &= \int \frac{x^2 + \frac{1}{x^2}}{x^3} dx \\ &= \int \left(\frac{1}{x} + \frac{1}{x^5} \right) dx = \ln|x| - \frac{1}{4x^4} + C.\end{aligned}$$

$$1639. \int \frac{x^2}{1+x^2} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{x^2}{1+x^2} dx &= \int \left(1 - \frac{1}{x^2+1} \right) dx \\ &= x - \operatorname{arctg} x + C.\end{aligned}$$

$$1640. \int \frac{x^2}{1-x^2} dx.$$

$$\text{解} \quad \int \frac{x^2}{1-x^2} dx = \int \left(-1 + \frac{1}{1-x^2} \right) dx$$

$$= -x + \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C.$$

$$1641. \int \frac{x^2 + 3}{x^2 - 1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2 + 3}{x^2 - 1} dx &= \int \left(1 + \frac{4}{x^2 - 1} \right) dx \\ &= x + 2 \ln \left| \frac{x-1}{x+1} \right| + C. \end{aligned}$$

$$1642. \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx \\ &= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}} \right) dx \\ &= \arcsin x + \ln(x + \sqrt{1+x^2}) + C. \end{aligned}$$

$$1643. \int \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^4-1}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^4-1}} dx \\ &= \int \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2+1}} \right) dx \\ &= \ln \left| \frac{x + \sqrt{x^2-1}}{x + \sqrt{x^2+1}} \right| + C. \end{aligned}$$

$$1644. \int (2^x + 3^x)^2 dx.$$

$$\begin{aligned} \text{解} \quad \int (2^x + 3^x)^2 dx &= \int (4^x + 2 \cdot 6^x + 9^x) dx \\ &= \frac{4^x}{\ln 4} + 2 \cdot \frac{6^x}{\ln 6} + \frac{9^x}{\ln 9} + C. \end{aligned}$$

$$1645. \int \frac{2^{x+1} - 5^{x+1}}{10^x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{2^{x+1} - 5^{x+1}}{10^x} dx &= \int \left[2 \left(\frac{1}{5} \right)^x - \frac{1}{5} \left(\frac{1}{2} \right)^x \right] dx \\ &= -\frac{2}{\ln 5} \left(\frac{1}{5} \right)^x + \frac{1}{5 \ln 2} \left(\frac{1}{2} \right)^x + C. \end{aligned}$$

$$1646. \int \frac{e^{3x} + 1}{e^x + 1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{e^{3x} + 1}{e^x + 1} dx &= \int (e^{2x} - e^x + 1) dx \\ &= \frac{1}{2} e^{2x} - e^x + x + C. \end{aligned}$$

$$1647. \int (1 + \sin x + \cos x) dx.$$

$$\text{解} \quad \int (1 + \sin x + \cos x) dx = x - \cos x + \sin x + C.$$

$$1648. \int \sqrt{1 - \sin 2x} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{1 - \sin 2x} dx &= \int \sqrt{(\cos x - \sin x)^2} dx \\ &= \int [\operatorname{sgn}(\cos x - \sin x)] (\cos x - \sin x) dx \\ &= (\sin x + \cos x) \cdot \operatorname{sgn}(\cos x - \sin x) + C. \end{aligned}$$

$$1649. \int \operatorname{ctg}^2 x dx.$$

$$\text{解} \quad \int \operatorname{ctg}^2 x dx = \int (\csc^2 x - 1) dx = -\operatorname{ctg} x - x + C.$$

$$1650. \int \operatorname{tg}^2 x dx.$$

$$\text{解} \quad \int \operatorname{tg}^2 x dx = \int (\sec^2 x - 1) dx = \operatorname{tg} x - x + C.$$

1651. $\int (a \operatorname{sh} x + b \operatorname{ch} x) dx.$

解 $\int (a \operatorname{sh} x + b \operatorname{ch} x) dx = a \operatorname{ch} x + b \operatorname{sh} x + C.$

1652. $\int \operatorname{th}^2 x dx.$

解 $\int \operatorname{th}^2 x dx = \int \left(1 - \frac{1}{\operatorname{ch}^2 x} \right) dx = x - \operatorname{th} x + C.$

1653. $\int \operatorname{cth}^2 x dx.$

解 $\int \operatorname{cth}^2 x dx = \int \left(1 + \frac{1}{\operatorname{sh}^2 x} \right) dx = x - \operatorname{cth} x + C.$

1654. 证明: 若

$$\int f(x) dx = F(x) + C,$$

则

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C \quad (a \neq 0).$$

证 由 $\int f(x) dx = F(x) + C$ 得知 $F'(x) = f(x)$. 因

而有 $F'(ax + b) = f(ax + b)$, 且 $\frac{d}{dx} \left[\frac{1}{a} F(ax + b) \right]$
 $= F'(ax + b)$, 于是

$$\frac{d}{dx} \left[\frac{1}{a} F(ax + b) \right] = f(ax + b),$$

所以

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C.$$

求出下列积分:

1655. $\int \frac{dx}{x+a}.$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\sqrt[5]{1-2x+x^2}}{1-x} dx \\
 &= \int (1-x)^{-\frac{3}{5}} dx \\
 &= -\frac{5}{2} \sqrt[5]{(1-x)^2} + C.
 \end{aligned}$$

$$1661. \int \frac{dx}{2+3x^2}.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{dx}{2+3x^2} = \int \frac{dx}{(\sqrt{2})^2 + (\sqrt{3}x)^2} \\
 &= \frac{1}{\sqrt{6}} \operatorname{arctg} \left[x \sqrt{\frac{3}{2}} \right] + C.
 \end{aligned}$$

$$1662. \int \frac{dx}{2-3x^2}.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{dx}{2-3x^2} = \frac{1}{2} \int \frac{dx}{1 - \left(\sqrt{\frac{3}{2}} x \right)^2} \\
 &= \frac{1}{2} \cdot \sqrt{\frac{2}{3}} \cdot \frac{1}{2} \ln \left| \frac{1 + \sqrt{\frac{3}{2}} x}{1 - \sqrt{\frac{3}{2}} x} \right| + C \\
 &= \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2} + x\sqrt{3}}{\sqrt{2} - x\sqrt{3}} \right| + C.
 \end{aligned}$$

$$1663. \int \frac{dx}{\sqrt{2-3x^2}}.$$

$$\text{解} \quad \int \frac{dx}{\sqrt{2-3x^2}} = \frac{1}{\sqrt{3}} \arcsin \left[x \sqrt{\frac{3}{2}} \right] + C.$$

$$1664. \int \frac{dx}{\sqrt{3x^2-2}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{dx}{\sqrt{3x^2-2}} &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left[\sqrt{\frac{3}{2}}x\right]^2-1}} \\
 &= \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2}{3}} \left| x \sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}x^2-1} \right| + C, \\
 &= \frac{1}{\sqrt{3}} \ln |x \sqrt{3} + \sqrt{3x^2-2}| + C.
 \end{aligned}$$

$$1665. \int (e^{-x} + e^{-2x}) dx.$$

$$\text{解} \quad \int (e^{-x} + e^{-2x}) dx = -(e^{-x} + \frac{1}{2}e^{-2x}) + C.$$

$$1666. \int (\sin 5x - \sin 5\alpha) dx.$$

$$\text{解} \quad \int (\sin 5x - \sin 5\alpha) dx = -\frac{1}{5} \cos 5x - x \sin 5\alpha + C.$$

$$1667. \int \frac{dx}{\sin^2\left(2x + \frac{\pi}{4}\right)}.$$

$$\text{解} \quad \int \frac{dx}{\sin^2\left(2x + \frac{\pi}{4}\right)} = -\frac{1}{2} \operatorname{ctg}\left(2x + \frac{\pi}{4}\right) + C.$$

$$1668. \int \frac{dx}{1 + \cos x}.$$

$$\text{解} \quad \int \frac{dx}{1 + \cos x} = \frac{1}{2} \int \frac{dx}{\cos^2 \frac{x}{2}} = \operatorname{tg} \frac{x}{2} + C.$$

$$1669. \int \frac{dx}{1 - \cos x}.$$

$$\text{解} \quad \int \frac{dx}{1 - \cos x} = \frac{1}{2} \int \frac{dx}{\sin^2 \frac{x}{2}} = -\operatorname{ctg} \frac{x}{2} + C.$$

$$1670. \int \frac{dx}{1+\sin x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{1+\sin x} &= \int \frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)} \\ &= -\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)+C. \end{aligned}$$

$$1671. \int [\operatorname{sh}(2x+1)+\operatorname{ch}(2x-1)]dx.$$

$$\begin{aligned} \text{解} \quad \int [\operatorname{sh}(2x+1)+\operatorname{ch}(2x-1)]dx \\ = \frac{1}{2} [\operatorname{ch}(2x+1)+\operatorname{sh}(2x-1)]+C. \end{aligned}$$

$$1672. \int \frac{dx}{\operatorname{ch}^2 \frac{x}{2}}.$$

$$\text{解} \quad \int \frac{dx}{\operatorname{ch}^2 \frac{x}{2}} = 2\operatorname{th} \frac{x}{2} + C.$$

$$1673. \int \frac{dx}{\operatorname{sh}^2 \frac{x}{2}}.$$

$$\text{解} \quad \int \frac{dx}{\operatorname{sh}^2 \frac{x}{2}} = -2\operatorname{cth} \frac{x}{2} + C.$$

用适当地变换被积函数的方法来求下列积分:

$$1674. \int \frac{xdx}{\sqrt{1-x^2}}.$$

$$\text{解} \quad \int \frac{xdx}{\sqrt{1-x^2}} = -\int \frac{d(1-x^2)}{2\sqrt{1-x^2}} = -\sqrt{1-x^2}+C.$$

$$1675. \int x^2 \sqrt[3]{1+x^3} dx.$$

$$\begin{aligned}\text{解} \quad \int x^2 \sqrt[3]{1+x^3} dx &= \frac{1}{3} \int (1+x^3)^{\frac{1}{3}} d(1+x^3) \\ &= \frac{1}{4} (1+x^3)^{\frac{4}{3}} + C.\end{aligned}$$

$$1676. \int \frac{x dx}{3-2x^2}.$$

$$\begin{aligned}\text{解} \quad \int \frac{x dx}{3-2x^2} &= -\frac{1}{4} \int \frac{d(3-2x^2)}{3-2x^2} \\ &= -\frac{1}{4} \ln |3-2x^2| + C.\end{aligned}$$

$$1677. \int \frac{x dx}{(1+x^2)^2}.$$

$$\begin{aligned}\text{解} \quad \int \frac{x dx}{(1+x^2)^2} &= \frac{1}{2} \int \frac{d(1+x^2)}{(1+x^2)^2} \\ &= -\frac{1}{2(1+x^2)} + C.\end{aligned}$$

$$1678. \int \frac{x dx}{4+x^4}.$$

$$\text{解} \quad \int \frac{x dx}{4+x^4} = \frac{1}{2} \int \frac{d(x^2)}{2^2 + (x^2)^2} = \frac{1}{4} \arctan \frac{x^2}{2} + C.$$

$$1679. \int \frac{x^3 dx}{x^8-2}.$$

$$\begin{aligned}\text{解} \quad \int \frac{x^3 dx}{x^8-2} &= \frac{1}{4} \int \frac{d(x^4)}{(x^4)^2 - (\sqrt{2})^2} \\ &= \frac{1}{8\sqrt{2}} \ln \left| \frac{x^4 - \sqrt{2}}{x^4 + \sqrt{2}} \right| + C.\end{aligned}$$

$$1680. \int \frac{dx}{\sqrt{x}(1+x)}.$$

$$\text{解} \quad \int \frac{dx}{\sqrt{x}(1+x)} = 2 \int \frac{d(\sqrt{x})}{1+(\sqrt{x})^2}$$

$$= 2 \operatorname{arc} \operatorname{tg} \sqrt{x} + C.$$

$$1681. \int \sin \frac{1}{x} \cdot \frac{dx}{x^2}.$$

$$\text{解} \quad \int \sin \frac{1}{x} \cdot \frac{dx}{x^2} = - \int \sin \frac{1}{x} d\left(\frac{1}{x}\right) = \cos \frac{1}{x} + C.$$

$$1682. \int \frac{dx}{x \sqrt{x^2 + 1}}.$$

$$\text{解} \quad \int \frac{dx}{x \sqrt{x^2 + 1}} = \int \frac{dx}{x|x| \sqrt{1 + \frac{1}{x^2}}}$$

$$= - \int \frac{d\left(\frac{1}{|x|}\right)}{\sqrt{1 + \left(\frac{1}{|x|}\right)^2}}$$

$$= - \ln \left[\frac{1}{|x|} + \sqrt{1 + \frac{1}{x^2}} \right] + C$$

$$= - \ln \left| \frac{1 + \sqrt{x^2 + 1}}{x} \right| + C.$$

$$1683. \int \frac{dx}{x \sqrt{x^2 - 1}}.$$

$$\text{解} \quad \int \frac{dx}{x \sqrt{x^2 - 1}} = \int \frac{dx}{x|x| \sqrt{1 - \frac{1}{x^2}}}$$

$$= - \int \frac{d\left(\frac{1}{|x|}\right)}{\sqrt{1 - \left(\frac{1}{|x|}\right)^2}} = - \operatorname{arc} \sin \frac{1}{|x|} + C.$$

$$1684. \int \frac{dx}{(x^2 + 1)^{\frac{3}{2}}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{dx}{(x^2 + 1)^{\frac{3}{2}}} &= \int \frac{\operatorname{sgn} x dx}{x^3 \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}}} \\
 &= -\frac{1}{2} \int \left(1 + \frac{1}{x^2}\right)^{-\frac{3}{2}} \operatorname{sgn} x d\left(1 + \frac{1}{x^2}\right) \\
 &= \left(1 + \frac{1}{x^2}\right)^{-\frac{1}{2}} \operatorname{sgn} x + C = \frac{x}{\sqrt{x^2 + 1}} + C.
 \end{aligned}$$

$$1685. \int \frac{x dx}{(x^2 - 1)^{\frac{3}{2}}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x dx}{(x^2 - 1)^{\frac{3}{2}}} &= \frac{1}{2} \int (x^2 - 1)^{-\frac{3}{2}} d(x^2 - 1) \\
 &= -\frac{1}{\sqrt{x^2 - 1}} + C.
 \end{aligned}$$

$$1686. \int \frac{x^2 dx}{(8x^3 + 27)^{\frac{2}{3}}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x^2 dx}{(8x^3 + 27)^{\frac{2}{3}}} &= \frac{1}{24} \int (8x^3 + 27)^{-\frac{2}{3}} d(8x^3 + 27) \\
 &= \frac{1}{8} \sqrt[3]{8x^3 + 27} + C.
 \end{aligned}$$

$$1687. \int \frac{dx}{\sqrt{x(1+x)}}.$$

解 由 $x(1+x) > 0$ 知: $x > 0$ 或 $x < -1$.

当 $x > 0$ 时,

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x(1+x)}} &= 2 \int \frac{d(\sqrt{x})}{\sqrt{1 + (\sqrt{x})^2}} \\
 &= 2 \ln(\sqrt{x} + \sqrt{1+x}) + C;
 \end{aligned}$$

当 $x < -1$ 时,

$$\begin{aligned}
& \int \frac{dx}{\sqrt{x(1+x)}} = - \int \frac{d[-(1+x)]}{\sqrt{(-x)[-(1+x)]}} \\
& = -2 \int \frac{d(\sqrt{-(1+x)})}{\sqrt{1+(\sqrt{-(1+x)})^2}} \\
& = -2 \ln(\sqrt{-x} + \sqrt{-(1+x)}) + C.
\end{aligned}$$

总之,得

$$\begin{aligned}
& \int \frac{dx}{\sqrt{x(1+x)}} \\
& = 2 \operatorname{sgn} x \cdot \ln(\sqrt{|x|} + \sqrt{|1+x|}) + C.
\end{aligned}$$

1688. $\int \frac{dx}{\sqrt{x(1-x)}}.$

解 由 $x(1-x) > 0$ 知; $0 < x < 1$. 于是,得

$$\begin{aligned}
& \int \frac{dx}{\sqrt{x(1-x)}} = 2 \int \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} \\
& = 2 \arcsin \sqrt{x} + C.
\end{aligned}$$

1689. $\int x e^{-x^2} dx.$

$$\begin{aligned}
\text{解} \quad \int x e^{-x^2} dx &= -\frac{1}{2} \int e^{-x^2} d(-x^2) \\
&= -\frac{1}{2} e^{-x^2} + C.
\end{aligned}$$

1690. $\int \frac{e^x dx}{2+e^x}.$

$$\text{解} \quad \int \frac{e^x dx}{2+e^x} = \int \frac{d(2+e^x)}{2+e^x} = \ln(2+e^x) + C.$$

1691. $\int \frac{dx}{e^x + e^{-x}}.$

$$\text{解} \quad \int \frac{dx}{e^x + e^{-x}} = \int \frac{d(e^x)}{1+(e^x)^2} = \arctan(e^x) + C.$$

$$1692. \int \frac{dx}{\sqrt{1+e^{2x}}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{1+e^{2x}}} &= - \int \frac{d(e^{-x})}{\sqrt{1+(e^{-x})^2}} \\ &= -\ln(e^{-x} + \sqrt{1+e^{-2x}}) + C. \end{aligned}$$

$$1693. \int \frac{\ln^2 x}{x} dx.$$

$$\text{解} \quad \int \frac{\ln^2 x}{x} dx = \int \ln^2 x d(\ln x) = \frac{1}{3} \ln^3 x + C.$$

$$1694. \int \frac{dx}{x \ln x \ln(\ln x)}$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x \ln x \ln(\ln x)} &= \int \frac{d(\ln x)}{\ln x \ln(\ln x)} \\ &= \int \frac{d[\ln(\ln x)]}{\ln(\ln x)} = \ln |\ln(\ln x)| + C. \end{aligned}$$

$$1695. \int \sin^5 x \cos x dx.$$

$$\text{解} \quad \int \sin^5 x \cos x dx = \int \sin^4 x d(\sin x) = \frac{1}{6} \sin^6 x + C.$$

$$1696. \int \frac{\sin x}{\sqrt{\cos^3 x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin x}{\sqrt{\cos^3 x}} dx &= - \int (\cos x)^{-\frac{3}{2}} d(\cos x) \\ &= \frac{2}{\sqrt{\cos x}} + C. \end{aligned}$$

$$1697. \int \operatorname{tg} x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{tg} x dx &= \int \frac{\sin x}{\cos x} dx = - \int \frac{d(\cos x)}{\cos x} \\ &= -\ln |\cos x| + C. \end{aligned}$$

$$1698. \int \operatorname{ctg} x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{ctg} x dx &= \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} \\ &= \ln |\sin x| + C. \end{aligned}$$

$$1699. \int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx.$$

$$\begin{aligned} \text{解} \quad &\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx \\ &= \int (\sin x - \cos x)^{-\frac{1}{3}} d(\sin x - \cos x) \\ &= \frac{3}{2} \sqrt[3]{(\sin x - \cos x)^2} + C = \frac{3}{2} \sqrt[3]{1 - \sin 2x} + C. \end{aligned}$$

$$1700. \int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx.$$

解 当 $|a| = |b| \neq 0$ 时,

$$\begin{aligned} &\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx \\ &= \frac{1}{|a|} \int \sin x \cos x dx = \frac{1}{2|a|} \sin^2 x + C; \end{aligned}$$

当 $|a| \neq |b|$ 时,

$$\begin{aligned} &\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx \\ &= \frac{1}{2} \int \frac{d(\sin^2 x)}{\sqrt{(a^2 - b^2) \sin^2 x + b^2}} \\ &= \frac{1}{a^2 - b^2} \sqrt{(a^2 - b^2) \sin^2 x + b^2} + C \\ &= \frac{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}}{a^2 - b^2} + C. \end{aligned}$$

$$1701. \int \frac{dx}{\sin^2 x \sqrt{\operatorname{ctg} x}}.$$

$$\begin{aligned}\text{解} \quad \int \frac{dx}{\operatorname{sh} x} &= \int \frac{\frac{1}{2\operatorname{ch}^2 \frac{x}{2}}}{\operatorname{th} \frac{x}{2}} dx = \int \frac{d\left(\operatorname{th} \frac{x}{2}\right)}{\operatorname{th} \frac{x}{2}} \\ &= \ln \left| \operatorname{th} \frac{x}{2} \right| + C.\end{aligned}$$

$$1706. \quad \int \frac{dx}{\operatorname{ch} x}.$$

$$\begin{aligned}\text{解} \quad \int \frac{dx}{\operatorname{ch} x} &= \int \frac{2dx}{e^x + e^{-x}} = 2 \int \frac{d(e^x)}{1 + (e^x)^2} \\ &= 2 \operatorname{arctg}(e^x) + C.\end{aligned}$$

$$1707. \quad \int \frac{\operatorname{sh} x \operatorname{ch} x}{\sqrt{\operatorname{sh}^4 x + \operatorname{ch}^4 x}} dx.$$

解 因为

$$\begin{aligned}\operatorname{sh}^4 x + \operatorname{ch}^4 x &= (\operatorname{sh}^2 x + \operatorname{ch}^2 x)^2 - 2\operatorname{sh}^2 x \operatorname{ch}^2 x \\ &= \operatorname{ch}^2 2x - \frac{1}{2} \operatorname{sh}^2 2x = \frac{1 + \operatorname{ch}^2 2x}{2},\end{aligned}$$

所以,得

$$\begin{aligned}\int \frac{\operatorname{sh} x \operatorname{ch} x}{\sqrt{\operatorname{sh}^4 x + \operatorname{ch}^4 x}} dx &= \int \frac{\frac{1}{4} d(\operatorname{ch} 2x)}{\frac{1}{\sqrt{2}} \sqrt{1 + \operatorname{ch}^2 2x}} \\ &= \frac{1}{2\sqrt{2}} \ln(\operatorname{ch} 2x + \sqrt{1 + \operatorname{ch}^2 2x}) + C_1 \\ &= \frac{1}{2\sqrt{2}} \ln\left(\frac{\operatorname{ch} 2x}{\sqrt{2}} + \sqrt{\operatorname{sh}^4 x + \operatorname{ch}^4 x}\right) + C.\end{aligned}$$

$$1708. \quad \int \frac{dx}{\operatorname{ch}^2 x \cdot \sqrt[3]{\operatorname{th}^2 x}};$$

$$\text{解} \quad \int \frac{dx}{\operatorname{ch}^2 x \cdot \sqrt[3]{\operatorname{th}^2 x}} = \int (\operatorname{th} x)^{-\frac{2}{3}} d(\operatorname{th} x)$$

$$= 3 \sqrt[3]{\operatorname{th} x} + C.$$

$$1709. \int \frac{\operatorname{arctg} x}{1+x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\operatorname{arctg} x}{1+x^2} dx &= \int \operatorname{arctg} x d(\operatorname{arctg} x) \\ &= \frac{1}{2} (\operatorname{arctg} x)^2 + C. \end{aligned}$$

$$1710. \int \frac{dx}{(\operatorname{arc} \sin x)^2 \sqrt{1-x^2}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{(\operatorname{arc} \sin x)^2 \sqrt{1-x^2}} &= \int \frac{d(\operatorname{arc} \sin x)}{(\operatorname{arc} \sin x)^2} \\ &= -\frac{1}{\operatorname{arc} \sin x} + C. \end{aligned}$$

$$1711. \int \sqrt{\frac{\ln(x + \sqrt{1+x^2})}{1+x^2}} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{\frac{\ln(x + \sqrt{1+x^2})}{1+x^2}} dx \\ &= \int [\ln(x + \sqrt{1+x^2})]^{\frac{1}{2}} d[\ln(x + \sqrt{1+x^2})] \\ &= \frac{2}{3} \ln^{\frac{3}{2}}(x + \sqrt{1+x^2}) + C. \end{aligned}$$

$$1712. \int \frac{x^2+1}{x^4+1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2+1}{x^4+1} dx &= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} \\ &= \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{x^2-1}{x\sqrt{2}} + C. \end{aligned}$$

$$1713. \int \frac{x^2-1}{x^4+1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2-1}{x^4+1} dx &= \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2} \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1} \right| + C. \end{aligned}$$

$$1714^+. \int \frac{x^{14} dx}{(x^5+1)^4}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^{14} dx}{(x^5+1)^4} &= \int \frac{x^{14} dx}{x^{20}(1+x^{-5})^4} \\ &= -\frac{1}{5} \int (1+x^{-5})^{-4} d(1+x^{-5}) \\ &= \frac{1}{15} (1+x^{-5})^{-3} + C_1 = \frac{x^{15}}{15(x^5+1)^3} + C_1 \\ &= \frac{(x^5+1)^3 - 3x^{10} - 3x^5 - 1}{15(x^5+1)^3} + C_1 \\ &= -\frac{3x^{10} + 3x^5 + 1}{15(x^5+1)^3} + C \end{aligned}$$

$$1715. \int \frac{x^{\frac{n}{2}} dx}{\sqrt{1+x^{n+2}}}.$$

解 当 $n = -2$ 时,

$$\int \frac{x^{\frac{n}{2}}}{\sqrt{1+x^{n+2}}} dx = \int \frac{dx}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \ln|x| + C;$$

当 $n \neq -2$ 时,

$$\begin{aligned} \int \frac{x^{\frac{n}{2}}}{\sqrt{1+x^{n+2}}} dx &= \frac{2}{n+2} \int \frac{d\left(x^{\frac{n+2}{2}}\right)}{\sqrt{1+\left(x^{\frac{n+2}{2}}\right)^2}} \\ &= \frac{2}{n+2} \ln\left(x^{\frac{n+2}{2}} + \sqrt{1+x^{n+2}}\right) + C. \end{aligned}$$

$$1716^+. \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx \\ &= \frac{1}{2} \int \ln \frac{1+x}{1-x} d \left(\ln \frac{1+x}{1-x} \right) \\ &= \frac{1}{4} \ln^2 \frac{1+x}{1-x} + C. \end{aligned}$$

$$1717. \int \frac{\cos x dx}{\sqrt{2+\cos 2x}}.$$

$$\begin{aligned} \text{解} \quad & \int \frac{\cos x dx}{\sqrt{2+\cos 2x}} = \int \frac{d(\sin x)}{\sqrt{3-2\sin^2 x}} \\ &= \frac{1}{\sqrt{2}} \arcsin \left[\sqrt{\frac{2}{3}} \sin x \right] + C. \end{aligned}$$

$$1718. \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \\ &= \frac{1}{2} \int \frac{\sin 2x dx}{1 - \frac{1}{2} \sin^2 2x} \\ &= -\frac{1}{4} \int \frac{d(\cos 2x)}{\frac{1+\cos^2 2x}{2}} \\ &= -\frac{1}{2} \arctg(\cos 2x) + C. \end{aligned}$$

$$1719. \int \frac{2^x \cdot 3^x}{9^x - 4^x} dx.$$

$$\text{解} \quad \int \frac{2^x \cdot 3^x}{9^x - 4^x} dx = \int \frac{\left(\frac{3}{2}\right)^x}{\left[\left(\frac{3}{2}\right)^x\right]^2 - 1} dx$$

$$\begin{aligned}
&= \frac{1}{\ln 3 - \ln 2} \int \frac{d\left[\left(\frac{3}{2}\right)^x\right]}{\left[\left(\frac{3}{2}\right)^x\right]^2 - 1} \\
&= \frac{1}{2(\ln 3 - \ln 2)} \ln \left| \frac{3^x - 2^x}{3^x + 2^x} \right| + C.
\end{aligned}$$

1720. $\int \frac{x dx}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}}.$

解
$$\begin{aligned}
&\int \frac{x dx}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} \\
&= \frac{1}{2} \int \frac{d(1+x^2)}{\sqrt{1+x^2} \cdot \sqrt{1+\sqrt{1+x^2}}} \\
&= \int \frac{d(1+\sqrt{1+x^2})}{\sqrt{1+\sqrt{1+x^2}}} \\
&= 2\sqrt{1+\sqrt{1+x^2}} + C.
\end{aligned}$$

用分项积分法计算下列积分：

1721. $\int x^2(2-3x^2)^2 dx.$

解
$$\begin{aligned}
\int x^2(2-3x^2)^2 dx &= \int (4x^2 - 12x^4 + 9x^6) dx \\
&= \frac{4}{3}x^3 - \frac{12}{5}x^5 + \frac{9}{7}x^7 + C.
\end{aligned}$$

1722. $\int \frac{1+x}{1-x} dx.$

解
$$\begin{aligned}
\int \frac{1+x}{1-x} dx &= \int \left(-1 + \frac{2}{1-x} \right) dx \\
&= -x - 2\ln|1-x| + C.
\end{aligned}$$

1723. $\int \frac{x^2}{1+x} dx.$

$$\begin{aligned}\text{解} \quad \int \frac{x^2}{1+x} dx &= \int \left(x - 1 + \frac{1}{1+x} \right) dx \\ &= \frac{1}{2}x^2 - x + \ln|1+x| + C.\end{aligned}$$

$$1724. \int \frac{x^3}{3+x} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{x^3}{3+x} dx &= \int \left(x^2 - 3x + 9 - \frac{27}{3+x} \right) dx \\ &= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln|3+x| + C.\end{aligned}$$

$$1725. \int \frac{(1+x)^2}{1+x^2} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{(1+x)^2}{1+x^2} dx &= \int \left(1 + \frac{2x}{1+x^2} \right) dx \\ &= x + \ln(1+x^2) + C.\end{aligned}$$

$$1726. \int \frac{(2-x)^2}{2-x^2} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{(2-x)^2}{2-x^2} dx &= \int \frac{(x^2-2)-4x+6}{2-x^2} dx \\ &= \int \left(-1 - \frac{4x}{2-x^2} + \frac{6}{2-x^2} \right) dx \\ &= -x + 2\ln|2-x^2| + \frac{3}{\sqrt{2}} \ln \left| \frac{\sqrt{2}+x}{\sqrt{2}-x} \right| + C.\end{aligned}$$

$$1727. \int \frac{x^2}{(1-x)^{100}} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{x^2}{(1-x)^{100}} dx &= \int \frac{(x-1+1)^2}{(1-x)^{100}} dx \\ &= \int [(1-x)^{-98} - 2(1-x)^{-99} + (1-x)^{-100}] dx \\ &= \frac{1}{97(1-x)^{97}} - \frac{1}{49(1-x)^{98}} + \frac{1}{99(1-x)^{99}} + C.\end{aligned}$$

$$1728. \int \frac{x^5}{x+1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^5}{x+1} dx &= \int \left(x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1} \right) dx \\ &= \frac{1}{5} x^5 - \frac{1}{4} x^4 + \frac{1}{3} x^3 - \frac{1}{2} x^2 + x - \ln |1+x| + C. \end{aligned}$$

$$1729. \int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}} &= \int \frac{1}{2} (\sqrt{x+1} - \sqrt{x-1}) dx \\ &= \frac{1}{3} \left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}} \right] + C. \end{aligned}$$

$$1730. \int x \sqrt{2-5x} dx.$$

$$\begin{aligned} \text{解} \quad \int x \sqrt{2-5x} dx &= \int \left[-\frac{1}{5} (2-5x) + \frac{2}{5} \right] (2-5x)^{\frac{1}{2}} dx \\ &= \int \left[-\frac{1}{5} (2-5x)^{\frac{3}{2}} + \frac{2}{5} (2-5x)^{\frac{1}{2}} \right] dx \\ &= \frac{2}{125} (2-5x)^{\frac{5}{2}} - \frac{4}{75} (2-5x)^{\frac{3}{2}} + C \\ &= -\frac{8+30x}{375} (2-5x)^{\frac{3}{2}} + C. \end{aligned}$$

$$1731. \int \frac{x dx}{\sqrt[3]{1-3x}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{\sqrt[3]{1-3x}} &= -\frac{1}{3} \int \frac{(1-3x) - 1}{(1-3x)^{\frac{1}{3}}} dx \\ &= -\frac{1}{3} \int \left[(1-3x)^{\frac{2}{3}} - (1-3x)^{-\frac{1}{3}} \right] dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{15} (1-3x)^{\frac{5}{3}} - \frac{1}{6} (1-3x)^{\frac{2}{3}} + C \\
&= -\frac{1+2x}{10} (1-3x)^{\frac{2}{3}} + C.
\end{aligned}$$

1732. $\int x^3 \sqrt[3]{1+x^2} dx.$

解
$$\begin{aligned}
&\int x^3 \sqrt[3]{1+x^2} dx \\
&= \frac{1}{2} \int [(x^2+1)-1] (1+x^2)^{\frac{1}{3}} d(1+x^2) \\
&= \frac{1}{2} \int [(1+x^2)^{\frac{4}{3}} - (1+x^2)^{\frac{1}{3}}] d(1+x^2) \\
&= \frac{3}{14} (1+x^2)^{\frac{7}{3}} - \frac{3}{8} (1+x^2)^{\frac{4}{3}} + C \\
&= \frac{12x^2-9}{56} (1+x^2)^{\frac{4}{3}} + C.
\end{aligned}$$

1733. $\int \frac{dx}{(x-1)(x+3)}.$

解
$$\begin{aligned}
&\int \frac{dx}{(x-1)(x+3)} = \frac{1}{4} \int \left(\frac{1}{x-1} - \frac{1}{x+3} \right) dx \\
&= \frac{1}{4} \ln \left| \frac{x-1}{x+3} \right| + C.
\end{aligned}$$

1734. $\int \frac{dx}{x^2+x-2}.$

解
$$\begin{aligned}
&\int \frac{dx}{x^2+x-2} = \frac{1}{3} \int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx \\
&= \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C.
\end{aligned}$$

1735. $\int \frac{dx}{(x^2+1)(x^2+2)}.$

解
$$\int \frac{dx}{(x^2+1)(x^2+2)} = \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2} \right) dx$$

$$= \operatorname{arc} \operatorname{tg} x - \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{x}{\sqrt{2}} + C.$$

$$1736. \int \frac{dx}{(x^2-2)(x^2+3)}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{(x^2-2)(x^2+3)} &= \frac{1}{5} \int \left(\frac{1}{x^2-2} - \frac{1}{x^2+3} \right) dx \\ &= \frac{1}{10\sqrt{2}} \ln \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| - \frac{1}{5\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} + C. \end{aligned}$$

$$1737. \int \frac{x dx}{(x+2)(x+3)}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{(x+2)(x+3)} &= \int \left(\frac{3}{x+3} - \frac{2}{x+2} \right) dx \\ &= \ln \frac{|x+3|^3}{(x+2)^2} + C. \end{aligned}$$

$$1738. \int \frac{x dx}{x^4+3x^2+2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{x^4+3x^2+2} &= \frac{1}{2} \int \frac{d(x^2)}{(x^2+1)(x^2+2)} \\ &= \frac{1}{2} \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2} \right) d(x^2) \\ &= \frac{1}{2} \ln \frac{x^2+1}{x^2+2} + C. \end{aligned}$$

$$1739. \int \frac{dx}{(x+a)^2(x+b)^2} \quad (a \neq b).$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{(x+a)^2(x+b)^2} &= \frac{1}{(a-b)^2} \int \left(\frac{1}{x+a} - \frac{1}{x+b} \right)^2 dx \\ &= \frac{1}{(a-b)^2} \int \left(\frac{1}{(x+a)^2} + \frac{1}{(x+b)^2} \right. \\ &\quad \left. - \frac{2}{(x+a)(x+b)} \right) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int [\cos \alpha - \cos(2x + \alpha)] dx \\
&= \frac{x}{2} \cos \alpha - \frac{1}{4} \sin(2x + \alpha) + C.
\end{aligned}$$

1744. $\int \sin 3x \cdot \sin 5x dx.$

解 $\int \sin 3x \cdot \sin 5x dx = \frac{1}{2} \int (\cos 2x - \cos 8x) dx$
 $= \frac{1}{4} \sin 2x - \frac{1}{16} \sin 8x + C.$

1745. $\int \cos \frac{x}{2} \cdot \cos \frac{x}{3} dx.$

解 $\int \cos \frac{x}{2} \cdot \cos \frac{x}{3} dx = \frac{1}{2} \int \left(\cos \frac{5x}{6} + \cos \frac{x}{6} \right) dx$
 $= \frac{3}{5} \sin \frac{5x}{6} + 3 \sin \frac{x}{6} + C.$

1746. $\int \sin \left(2x - \frac{\pi}{6} \right) \cdot \cos \left(3x + \frac{\pi}{4} \right) dx.$

解 $\int \sin \left(2x - \frac{\pi}{6} \right) \cdot \cos \left(3x + \frac{\pi}{4} \right) dx$
 $= \frac{1}{2} \int \left[\sin \left(5x + \frac{\pi}{12} \right) - \sin \left(x + \frac{5\pi}{12} \right) \right] dx$
 $= -\frac{1}{10} \cos \left(5x + \frac{\pi}{12} \right) + \frac{1}{2} \cos \left(x + \frac{5\pi}{12} \right) + C.$

1747. $\int \sin^3 x dx.$

解 $\int \sin^3 x dx = \int (\cos^2 x - 1) d(\cos x)$
 $= \frac{1}{3} \cos^3 x - \cos x + C.$

1748. $\int \cos^3 x dx.$

解 $\int \cos^3 x dx = \int (1 - \sin^2 x) d(\sin x)$

$$= \sin x - \frac{1}{3} \sin^3 x + C.$$

1749. $\int \sin^4 x dx.$

$$\begin{aligned} \text{解} \quad \int \sin^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\ &= \frac{1}{8} \int (3 - 4\cos 2x + \cos 4x) dx \\ &= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C. \end{aligned}$$

1750. $\int \cos^4 x dx.$

$$\begin{aligned} \text{解} \quad \int \cos^4 x dx &= \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\ &= \frac{1}{8} \int (3 + 4\cos 2x + \cos 4x) dx \\ &= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C. \end{aligned}$$

1751. $\int \operatorname{ctg}^2 x dx.$

$$\text{解} \quad \int \operatorname{ctg}^2 x dx = \int (\csc^2 x - 1) dx = -\operatorname{ctg} x - x + C.$$

1752. $\int \operatorname{tg}^3 x dx.$

$$\begin{aligned} \text{解} \quad \int \operatorname{tg}^3 x dx &= \int \operatorname{tg} x \cdot (\sec^2 x - 1) dx \\ &= \int \operatorname{tg} x d(\operatorname{tg} x) - \int \operatorname{tg} x dx \\ &= \frac{1}{2} \operatorname{tg}^2 x + \ln |\cos x| + C, \end{aligned}$$

其中第二个积分见 1697 题.

1753. $\int \sin^2 3x \cdot \sin^3 2x dx.$

解 因为

$$\begin{aligned}\sin^2 3x \cdot \sin^3 2x &= \frac{1}{2}(1 - \cos 6x) \cdot \frac{1}{4}(3\sin 2x - \sin 6x) \\&= \frac{1}{8}(3\sin 2x - 3\cos 6x \cdot \sin 2x - \sin 6x \\&\quad + \sin 6x \cdot \cos 6x) \\&= \frac{3}{8}\sin 2x + \frac{3}{16}\sin 4x - \frac{1}{8}\sin 6x - \frac{3}{16}\sin 8x \\&\quad + \frac{1}{16}\sin 12x\end{aligned}$$

所以,得

$$\begin{aligned}\int \sin^2 3x \cdot \sin^3 2x dx &= -\frac{3}{16}\cos 2x - \frac{3}{64}\cos 4x \\&\quad + \frac{1}{48}\cos 6x + \frac{3}{128}\cos 8x - \frac{1}{192}\cos 12x + C.\end{aligned}$$

1754. $\int \frac{dx}{\sin^2 x \cdot \cos^2 x}.$

解 $\int \frac{dx}{\sin^2 x \cdot \cos^2 x} = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right) dx$
 $= -\operatorname{ctg} x + \operatorname{tg} x + C.$

1755. $\int \frac{dx}{\sin^2 x \cdot \cos x}.$

解 $\int \frac{dx}{\sin^2 x \cdot \cos x} = \int \left(\frac{1}{\cos x} + \frac{\cos x}{\sin^2 x} \right) dx$
 $= \int \frac{dx}{\cos x} + \int \frac{d(\sin x)}{\sin^2 x}$
 $= \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| - \frac{1}{\sin x} + C,$

其中第一个积分见 1704 题.

$$1756. \int \frac{dx}{\sin x \cdot \cos^3 x}.$$

$$\begin{aligned}\text{解} \quad \int \frac{dx}{\sin x \cdot \cos^3 x} &= \int \left(\frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x} \right) dx \\ &= - \int \frac{d(\cos x)}{\cos^3 x} + \int \frac{d(2x)}{\sin 2x} \\ &= \frac{1}{2\cos^2 x} + \ln |\operatorname{tg} x| + C,\end{aligned}$$

其中第二个积分见 1703 题.

$$1757. \int \frac{\cos^3 x}{\sin x} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{\cos^3 x}{\sin x} dx &= \int \frac{1 - \sin^2 x}{\sin x} \cos x dx \\ &= \int \left(\frac{1}{\sin x} - \sin x \right) d(\sin x) \\ &= \ln |\sin x| - \frac{1}{2} \sin^2 x + C.\end{aligned}$$

$$1758. \int \frac{dx}{\cos^4 x}.$$

$$\begin{aligned}\text{解} \quad \int \frac{dx}{\cos^4 x} &= \int \sec^2 x \cdot \frac{dx}{\cos^2 x} = \int (1 + \operatorname{tg}^2 x) d(\operatorname{tg} x) \\ &= \operatorname{tg} x + \frac{1}{3} \operatorname{tg}^3 x + C.\end{aligned}$$

$$1759. \int \frac{dx}{1+e^x}.$$

$$\text{解} \quad \int \frac{dx}{1+e^x} = \int \left(1 - \frac{e^x}{1+e^x} \right) dx = x - \ln(1+e^x) + C.$$

$$1760. \int \frac{(1+e^x)^2}{1+e^{2x}} dx.$$

$$\text{解} \quad \int \frac{(1+e^x)^2}{1+e^{2x}} dx = \int \left(1 + \frac{2e^x}{1+e^{2x}} \right) dx$$

$$=x+2\operatorname{arc\,tg}(e^x)+C.$$

$$1761. \int \operatorname{sh}^2 x dx.$$

$$\text{解} \quad \int \operatorname{sh}^2 x dx = \int \frac{\operatorname{ch} 2x + 1}{2} dx = \frac{1}{4} \operatorname{sh} 2x + \frac{x}{2} + C.$$

$$1762. \int \operatorname{ch}^2 x dx.$$

$$\text{解} \quad \int \operatorname{ch}^2 x dx = \int \frac{\operatorname{ch} 2x + 1}{2} dx = \frac{1}{4} \operatorname{sh} 2x + \frac{x}{2} + C.$$

$$1763. \int \operatorname{sh} x \cdot \operatorname{sh} 2x dx.$$

$$\text{解} \quad \int \operatorname{sh} x \cdot \operatorname{sh} 2x dx = 2 \int \operatorname{sh}^2 x \operatorname{ch} x dx = 2 \int \operatorname{sh}^2 x d(\operatorname{sh} x)$$

$$= \frac{2}{3} \operatorname{sh}^3 x + C.$$

$$1764. \int \operatorname{ch} x \cdot \operatorname{ch} 3x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{ch} x \cdot \operatorname{ch} 3x dx &= \frac{1}{2} \int (\operatorname{ch} 4x + \operatorname{ch} 2x) dx \\ &= \frac{1}{8} \operatorname{sh} 4x + \frac{1}{4} \operatorname{sh} 2x + C. \end{aligned}$$

$$1765. \int \frac{dx}{\operatorname{sh}^2 x \cdot \operatorname{ch}^2 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\operatorname{sh}^2 x \cdot \operatorname{ch}^2 x} &= \int \left(\frac{1}{\operatorname{sh}^2 x} - \frac{1}{\operatorname{ch}^2 x} \right) dx \\ &= -(\operatorname{cth} x + \operatorname{th} x) + C. \end{aligned}$$

用适当的代换,求下列积分:

$$1766. \int x^2 \sqrt[3]{1-x} dx.$$

解 设 $1-x=t$, 则 $x=1-t$, $dx=-dt$, 故得

$$\begin{aligned}
& \int x^2 \sqrt[3]{1-x} dx = - \int (1-t)^2 t^{\frac{1}{3}} dt \\
& = - \int \left(t^{\frac{1}{3}} - 2t^{\frac{4}{3}} + t^{\frac{7}{3}} \right) dt \\
& = - \frac{3}{4} t^{\frac{4}{3}} + \frac{6}{7} t^{\frac{7}{3}} - \frac{3}{10} t^{\frac{10}{3}} + C \\
& = - \frac{3}{140} (9 + 12x + 14x^2) (1-x)^{\frac{4}{3}} + C.
\end{aligned}$$

1767. $\int x^3 (1-5x^2)^{10} dx.$

解 设 $1-5x^2=t$, 则 $x^2=\frac{1}{5}(1-t)$, 从而 $x^3 dx$

$$\begin{aligned}
& = \frac{1}{2} x^2 d(x^2) = \frac{1}{10} (1-t) \left(-\frac{1}{5} \right) dt \\
& = -\frac{1}{50} (1-t) dt, \text{故得} \\
& \int x^3 (1-5x^2)^{10} dx = -\frac{1}{50} \int (t^{10} - t^{11}) dt \\
& = -\frac{1}{550} t^{11} + \frac{1}{600} t^{12} + C \\
& = -\frac{1+55x^2}{6600} (1-5x^2)^{11} + C.
\end{aligned}$$

1768. $\int \frac{x^2}{\sqrt{2-x}} dx.$

解 设 $2-x=t$, 则 $x=2-t$, $dx=-dt$, 故得

$$\begin{aligned}
& \int \frac{x^2}{\sqrt{2-x}} dx = - \int t^{-\frac{1}{2}} (2-t)^2 dt \\
& = - \int \left(4t^{-\frac{1}{2}} - 4t^{\frac{1}{2}} + t^{\frac{3}{2}} \right) dt \\
& = -8t^{\frac{1}{2}} + \frac{8}{3} t^{\frac{3}{2}} - \frac{2}{5} t^{\frac{5}{2}} + C \\
& = -\frac{2}{15} (32 + 8x + 3x^2) \sqrt{2-x} + C.
\end{aligned}$$

$$1769. \int \frac{x^5}{\sqrt{1-x^2}} dx.$$

解 设 $1-x^2=t$, 则 $x^2=1-t$, 从而 $x^5 dx = \frac{1}{2}(x^2)^2$

$\cdot d(x^2) = -\frac{1}{2}(1-t)^2 dt$, 故得

$$\begin{aligned} \int \frac{x^5}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int t^{-\frac{1}{2}} (1-t)^2 dt \\ &= -\frac{1}{2} \int (t^{-\frac{1}{2}} - 2t^{\frac{1}{2}} + t^{\frac{3}{2}}) dt \\ &= -t^{\frac{1}{2}} + \frac{2}{3}t^{\frac{3}{2}} - \frac{1}{5}t^{\frac{5}{2}} + C \\ &= -\frac{1}{15}(8+4x^2+3x^4)\sqrt{1-x^2} + C. \end{aligned}$$

$$1770. \int x^5 (2-5x^3)^{\frac{2}{3}} dx.$$

解 设 $2-5x^3=t$, 则 $x^3=\frac{1}{5}(2-t)$, 从而

$$x^5 dx = \frac{1}{3}x^3 d(x^3) = -\frac{1}{75}(2-t)dt,$$

故得

$$\begin{aligned} \int x^5 (2-5x^3)^{\frac{2}{3}} dx &= -\frac{1}{75} \int t^{\frac{2}{3}} (2-t) dt \\ &= -\frac{1}{75} \int (2t^{\frac{2}{3}} - t^{\frac{5}{3}}) dt = -\frac{2}{125}t^{\frac{5}{3}} + \frac{1}{200}t^{\frac{8}{3}} + C \\ &= -\frac{6+25x^3}{1000}(2-5x^3)^{\frac{5}{3}} + C. \end{aligned}$$

$$1771^+. \int \cos^5 x \sqrt{\sin x} dx.$$

解 设 $\sin x=t$, 则 $\cos^5 x dx = (1-\sin^2 x)^2 d(\sin x)$
 $= (1-t^2)^2 dt,$

故得

$$\begin{aligned}
& \int \cos^5 x \sqrt{\sin x} dx = \int (1-t^2)^2 t^{\frac{1}{2}} dt \\
& = \int \left(t^{\frac{1}{2}} - 2t^{\frac{5}{2}} + t^{\frac{9}{2}} \right) dt \\
& = \frac{2}{3} t^{\frac{3}{2}} - \frac{4}{7} t^{\frac{7}{2}} + \frac{2}{11} t^{\frac{11}{2}} + C \\
& = \left(\frac{2}{3} - \frac{4}{7} \sin^2 x + \frac{2}{11} \sin^4 x \right) \sqrt{\sin^3 x} + C.
\end{aligned}$$

1772. $\int \frac{\sin x \cos^3 x}{1 + \cos^2 x} dx.$

解 设 $\cos^2 x = t$, 则 $\sin x \cos x dx = -\frac{1}{2} dt$, 故得

$$\begin{aligned}
& \int \frac{\sin x \cos^3 x}{1 + \cos^2 x} dx = -\frac{1}{2} \int \frac{t}{1+t} dt \\
& = -\frac{1}{2} \int \left(1 - \frac{1}{1+t} \right) dt = -\frac{1}{2} t + \frac{1}{2} \ln(1+t) + C \\
& = -\frac{1}{2} \cos^2 x + \frac{1}{2} \ln(1 + \cos^2 x) + C.
\end{aligned}$$

1773. $\int \frac{\sin^2 x}{\cos^6 x} dx.$

解 设 $\operatorname{tg} x = t$, 则 $\frac{1}{\cos^4 x} dx = (1+t^2) dt$, 故得

$$\begin{aligned}
& \int \frac{\sin^2 x}{\cos^6 x} dx = \int (t^4 + t^2) dt = \frac{1}{5} t^5 + \frac{1}{3} t^3 + C \\
& = \frac{1}{5} \operatorname{tg}^5 x + \frac{1}{3} \operatorname{tg}^3 x + C.
\end{aligned}$$

1774. $\int \frac{\ln x dx}{x \sqrt{1 + \ln x}}.$

解 设 $1 + \ln x = t$, 则 $\frac{\ln x dx}{x}$

$$= (1 + \ln x - 1) d(1 + \ln x)$$

$$= (t - 1) dt, \text{ 故得}$$

$$\begin{aligned}
& \int \frac{\ln x dx}{x \sqrt{1+\ln x}} = \int t^{-\frac{1}{2}} (t-1) dt \\
& = \int \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) dt = \frac{2}{3} t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + C \\
& = \frac{2}{3} (\ln x - 2) \sqrt{1+\ln x} + C.
\end{aligned}$$

1775. $\int \frac{dx}{e^{\frac{x}{2}} + e^x}.$

解 设 $e^{\frac{x}{2}} = t$, 则 $e^x = t^2, dx = \frac{2dt}{t}$, 故得

$$\begin{aligned}
& \int \frac{dx}{e^{\frac{x}{2}} + e^x} = 2 \int \frac{dt}{t^2(1+t)} = 2 \int \left(\frac{1-t}{t^2} + \frac{1}{1+t} \right) dt \\
& = -\frac{2}{t} - 2\ln t + 2\ln(1+t) + C \\
& = -2e^{-\frac{x}{2}} - x + 2\ln(1+e^{\frac{x}{2}}) + C.
\end{aligned}$$

1776. $\int \frac{dx}{\sqrt{1+e^x}}.$

解 设 $\sqrt{1+e^x} = t$, 则 $x = \ln(t^2 - 1), dx = \frac{2t}{t^2 - 1} dt$,

故得

$$\begin{aligned}
& \int \frac{dx}{\sqrt{1+e^x}} \\
& = 2 \int \frac{dt}{t^2 - 1} = \ln \left(\frac{t-1}{t+1} \right) + C \\
& = \ln \left[\frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} \right] + C \\
& = x - 2\ln(1 + \sqrt{1+e^x}) + C.
\end{aligned}$$

1777. $\int \frac{\operatorname{arc\,tg} \sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x}.$

代入得

$$\begin{aligned}
 \int \frac{x^2 dx}{\sqrt{x^2-2}} &= 2 \int \sec^3 t dt = 2 \int \frac{d(\sin t)}{(1-\sin^2 t)^2} \\
 &= \frac{1}{2} \int \left(\frac{1}{1+\sin t} + \frac{1}{1-\sin t} \right)^2 d(\sin t) \\
 &= \frac{1}{2} \int \frac{d(1+\sin t)}{(1+\sin t)^2} - \frac{1}{2} \int \frac{d(1-\sin t)}{(1-\sin t)^2} \\
 &\quad + \int \frac{d(\sin t)}{1-\sin^2 t} \\
 &= \frac{1}{2} \left(\frac{1}{1-\sin t} - \frac{1}{1+\sin t} \right) + \frac{1}{2} \ln \left(\frac{1+\sin t}{1-\sin t} \right) + C_1 \\
 &= \operatorname{tg} t \cdot \sec t + \ln(\sec t + \operatorname{tg} t) + C_1 \\
 &= \frac{x}{2} \sqrt{x^2-2} + \ln(x + \sqrt{x^2-2}) + C.
 \end{aligned}$$

(2) 当 $x < -\sqrt{2}$ 时, 仍设 $x = \sqrt{2} \sec t$, 但限制 $\pi < t < \frac{3\pi}{2}$. 其余步骤与上相同, 注意到, 此时 $\sec t + \operatorname{tg} t < 0$, 因此在对数符号里要加绝对值, 即结果为

$$\frac{x}{2} \sqrt{x^2-2} + \ln |x + \sqrt{x^2-2}| + C.$$

总之, 当 $|x| > \sqrt{2}$ 时,

$$\int \frac{x^2 dx}{\sqrt{x^2-2}} = \frac{x}{2} \sqrt{x^2-2} + \ln |x + \sqrt{x^2-2}| + C.$$

1780. $\int \sqrt{a^2-x^2} dx.$

解 被积函数的存在域为 $-a \leq x \leq a$, 因此设 $x = a \sin t$, 并限制 $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. 从而

$$\sqrt{a^2-x^2} = a \cos t, dx = a \cos t dt.$$

代入得

$$\begin{aligned}\int \sqrt{a^2 - x^2} dx &= a^2 \int \cos^2 t dt \\&= a^2 \left(\frac{t}{2} + \frac{1}{4} \sin 2t \right) + C \\&= \frac{a^2}{2} t + \frac{a^2}{2} \sin t \cos t + C \\&= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.\end{aligned}$$

*) 利用 1742 题的结果.

1781. $\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}}.$

解 被积函数的存在域为 $-\infty < x < +\infty$, 因此可设

$x = a \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$. 从而

$$(x^2 + a^2)^{\frac{3}{2}} = a^3 \sec^3 t, dx = a \sec^2 t dt.$$

代入得

$$\begin{aligned}\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}} &= \frac{1}{a^2} \int \cos t dt = \frac{1}{a^2} \sin t + C \\&= \frac{1}{a^2} \cdot \frac{\operatorname{tg} t}{\sqrt{1 + \operatorname{tg}^2 t}} + C = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.\end{aligned}$$

1782. $\int \sqrt{\frac{a+x}{a-x}} dx.$

解 被积函数的存在域为 $-a \leq x < a$, 因此可设 $x =$

$a \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$. 从而

$$\sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{1+\sin t}{1-\sin t}} = \frac{1+\sin t}{\cos t}, dx = a \cos t dt.$$

代入得

$$\begin{aligned}
& \int \sqrt{\frac{a+x}{a-x}} dx \\
&= a \int (1 + \sin t) dt = a(t - \cos t) + C \\
&= a \arcsin \frac{x}{a} - \sqrt{a^2 - x^2} + C \quad (-a < x < a).
\end{aligned}$$

注意, 上式在端点 $x = -a$ 也成立. 即函数 $F(x) = a \arcsin \frac{x}{a} - \sqrt{a^2 - x^2}$ 在点 $x = -a$ 的(右)导数等于被

积函数 $f(x) = \sqrt{\frac{a+x}{a-x}}$ 在点 $x = -a$ 之值. 事实上, 由于 $F(x)$ 和 $f(x)$ 都在 $-a \leq x < a$ 连续, 且 $F'(x) = f(x)$ 在 $-a < x < a$ 成立. 故由中值定理, 知当 $-a < x < a$ 时, 有

$$\frac{F(x) - F(-a)}{x + a} = F'(\xi) = f(\xi), \quad -a < \xi < x.$$

由此可知, (右)导数

$$\begin{aligned}
F'(-a) &= \lim_{x \rightarrow -a+0} \frac{F(x) - F(-a)}{x + a} \\
&= \lim_{\xi \rightarrow -a+0} f(\xi) = f(-a).
\end{aligned}$$

下面有些题目在端点的情况可类似地进行讨论, 从略.

1783. $\int x \sqrt{\frac{x}{2a-x}} dx.$

解 被积函数的存在域为 $0 \leq x < 2a$, 因此可设

$x = 2a \sin^2 t$, 并限制 $0 \leq t < \frac{\pi}{2}$. 从而

$$x \sqrt{\frac{x}{2a-x}} = \frac{2a \sin^3 t}{\cos t}, \quad dx = 4a \sin t \cos t dt.$$

代入得

$$\begin{aligned} & \int x \sqrt{\frac{x}{2a-x}} \\ &= 8a^2 \int \sin^4 t dt \\ &= 8a^2 \left(\frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t \right) + C. \end{aligned}$$

注意到 $\sin 2t = 2\sin t \cos t = 2 \sqrt{\frac{x}{2a}} \cdot \sqrt{1 - \frac{x}{2a}} = \frac{1}{a} \cdot \sqrt{x(2a-x)}$ 及 $\sin 4t = 2\sin 2t \cos 2t = 4\sin t \cos t (1 - 2\sin^2 t) = \frac{2}{a^2}(a-x)\sqrt{x(2a-x)}$, 最后得

$$\begin{aligned} \int x \sqrt{\frac{x}{2a-x}} dx &= 3a^2 \arcsin \sqrt{\frac{x}{2a}} - 2a^2 \cdot \\ & \frac{1}{a} \sqrt{x(2a-x)} + \frac{1}{4}a^2 \cdot \frac{2}{a^2}(a-x)\sqrt{x(2a-x)} + C \\ &= 3a^2 \arcsin \sqrt{\frac{x}{2a}} - \frac{3a+x}{2} \sqrt{x(2a-x)} + C. \end{aligned}$$

*) 利用 1749 题的结果.

1784. $\int \frac{dx}{\sqrt{(x-a)(b-x)}}.$

解 不妨设 $a < b$. 被积函数的存在域为 $a < x < b$,

因此可设 $x-a = (b-a)\sin^2 t$, 并限制 $0 < t < \frac{\pi}{2}$. 从而

$$\sqrt{(x-a)(b-x)} = (b-a)\sin t \cos t,$$

$$dx = 2(b-a)\sin t \cos t dt.$$

代入得

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \int dt = 2t + C$$

$$= 2 \arcsin \sqrt{\frac{x-a}{b-a}} + C.$$

1785. $\int \sqrt{(x-a)(b-x)} dx.$

解 与上题相同, 作同一代换, 并注意到 $\sin 4t = 4 \sin t$

$$\cdot \cos t (1 - 2 \sin^2 t) = 4 \sqrt{\frac{x-a}{b-a}}.$$

$$\begin{aligned} & \sqrt{1 - \frac{x-a}{b-a}} \left(1 - 2 \cdot \frac{x-a}{b-a} \right) \\ &= -4 \cdot \frac{2x - (a+b)}{(b-a)^2} \sqrt{(x-a)(b-x)}, \text{ 即得} \end{aligned}$$

$$\begin{aligned} \int \sqrt{(x-a)(b-x)} dx &= 2(b-a)^2 \int \sin^2 t \cos^2 t dt \\ &= \frac{(b-a)^2}{2} \int \sin^2 2t dt = \frac{(b-a)^2}{4} \int (1 - \cos 4t) dt \\ &= \frac{(b-a)^2}{4} \left(t - \frac{1}{4} \sin 4t \right) + C \\ &= \frac{(b-a)^2}{4} \arcsin \sqrt{\frac{x-a}{b-a}} \\ &\quad + \frac{2x - (a+b)}{4} \sqrt{(x-a)(b-x)} + C. \end{aligned}$$

用双曲线代换 $x = a \operatorname{sh} t, x = a \operatorname{ch} t$ 等等, 求下列积分 (参数为正的):

1786. $\int \sqrt{a^2 + x^2} dx.$

解 被积函数的存在域为 $-\infty < x < +\infty$, 因此可设 $x = a \operatorname{sh} t$. 从而

$$\sqrt{a^2 + x^2} = a \operatorname{ch} t, dx = a \operatorname{ch} t dt.$$

代入得

$$\int \sqrt{a^2 + x^2} dx = a^2 \int \operatorname{ch}^2 t dt$$

$$= a^2 \left(\frac{t}{2} + \frac{1}{4} \operatorname{sh} 2t \right) + C_1.$$

注意到 $x + \sqrt{a^2 + x^2} = a(\operatorname{sh} t + \operatorname{ch} t) = ae^t$, 即 $t = \ln \frac{x + \sqrt{a^2 + x^2}}{a}$ 及 $\operatorname{sh} 2t = 2\operatorname{sh} t \operatorname{ch} t = \frac{2x \sqrt{a^2 + x^2}}{a^2}$,

最后得

$$\int \sqrt{a^2 + x^2} dx = \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2})$$

$$+ \frac{x}{2} \sqrt{a^2 + x^2} + C.$$

*) 利用 1762 题的结果.

1787. $\int \frac{x^2}{\sqrt{a^2 + x^2}} dx.$

解 与上题相同, 设 $x = a \operatorname{sh} t$, 则

$$\frac{x^2}{\sqrt{a^2 + x^2}} = \frac{a \operatorname{sh}^2 t}{\operatorname{ch} t}, dx = a \operatorname{ch} t dt.$$

代入得

$$\int \frac{x^2}{\sqrt{a^2 + x^2}} dx = a^2 \int \operatorname{sh}^2 t dt$$

$$= a^2 \left(\frac{1}{4} \operatorname{sh} 2t - \frac{t}{2} \right) + C_1$$

$$= \frac{x}{2} \sqrt{a^2 + x^2} - \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C.$$

*) 利用 1761 题的结果.

1788⁺. $\int \sqrt{\frac{x-a}{x+a}} dx.$

解 被积函数的存在域为 $x \geq a$ 及 $x < -a$.

(1) 当 $x > a$ 时, 可设 $x = a \cosh t$, 并限制 $t > 0$.
从而

$$\sqrt{\frac{x-a}{x+a}} = \frac{\cosh t - 1}{\sinh t}, dx = a \sinh t dt.$$

代入得

$$\begin{aligned} \int \sqrt{\frac{x-a}{x+a}} dx &= a \int (\cosh t - 1) dt \\ &= a \sinh t - at + C_1 = a \sqrt{\cosh^2 t - 1} - at + C_1 \\ &= a \sqrt{\left(\frac{x}{a}\right)^2 - 1} - a \ln \left[\sqrt{\left(\frac{x}{a}\right)^2 - 1} + \frac{x}{a} \right] + C_1 \\ &= \sqrt{x^2 - a^2} - a \ln(\sqrt{x^2 - a^2} + x) + C_2 \\ &= \sqrt{x^2 - a^2} - 2a \ln(\sqrt{x - a} + \sqrt{x + a}) + C. \end{aligned}$$

(2) 当 $x < -a$ 时, 可设 $x = -a \cosh t$, 并限制 $t > 0$.
从而

$$\sqrt{\frac{x-a}{x+a}} = \frac{\cosh t + 1}{\sinh t}, dx = -a \sinh t dt.$$

代入得

$$\begin{aligned} \int \sqrt{\frac{x-a}{x+a}} dx &= -a \int (\cosh t + 1) dt \\ &= -a \sinh t - at + C_1 \\ &= -a \cdot \sqrt{\left(\frac{x}{a}\right)^2 - 1} \\ &\quad - a \ln \left[\sqrt{\left(\frac{x}{a}\right)^2 - 1} - \frac{x}{a} \right] + C_1 \\ &= -\sqrt{x^2 - a^2} - a \ln(\sqrt{x^2 - a^2} - x) + C_2 \end{aligned}$$

$$= -\sqrt{x^2 - a^2} - 2a \ln(\sqrt{-x + a} + \sqrt{-x - a}) + C.$$

总之,当 $|x| > a$ 时,

$$\int \sqrt{\frac{x-a}{x+a}} dx = \operatorname{sgn} x \cdot \sqrt{x^2 - a^2} - 2a \ln(\sqrt{|x-a|} + \sqrt{|x+a|}) + C.$$

1789. $\int \frac{dx}{\sqrt{(x+a)(x+b)}}.$

解 不妨设 $a < b$. 被积函数的存在域为 $x > -a$ 及 $x < -b$.

(1) 当 $x > -a$ 时,可设 $x+a = (b-a)\operatorname{sh}^2 t$,并限制 $t > 0$. 从而

$$\begin{aligned} \sqrt{(x+a)(x+b)} &= (b-a)\operatorname{sh} t \operatorname{ch} t, dx \\ &= 2(b-a)\operatorname{sh} t \operatorname{ch} t dt. \end{aligned}$$

代入得

$$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = 2 \int dt = 2t + C_1.$$

注意到 $\sqrt{x+a} + \sqrt{x+b} = \sqrt{b-a}(\operatorname{sh} t + \operatorname{ch} t) =$

$\sqrt{b-a}e^t$, 就有 $t = \ln \frac{\sqrt{x+a} + \sqrt{x+b}}{\sqrt{b-a}}$, 最后得

$$\begin{aligned} \int \frac{dx}{\sqrt{(x+a)(x+b)}} \\ = 2 \ln(\sqrt{x+a} + \sqrt{x+b}) + C. \end{aligned}$$

(2) 当 $x < -b$ 时,可设 $x+b = (a-b)\operatorname{sh}^2 t$,并限制 $t > 0$. 从而

$$\begin{aligned}\sqrt{(x+a)(x+b)} &= (b-a)\operatorname{sh}t \operatorname{ch}t, dx \\ &= -(b-a)2\operatorname{sh}t \operatorname{ch}t dt.\end{aligned}$$

代入得

$$\begin{aligned}\int \frac{dx}{\sqrt{(x+a)(x+b)}} &= -2 \int dt = -2t + C_1 \\ &= -2\ln(\sqrt{-(x+a)} + \sqrt{-(x+b)}) + C.\end{aligned}$$

总之,

$$\begin{aligned}&\int \frac{dx}{\sqrt{(x+a)(x+b)}} \\ &= \begin{cases} 2\ln(\sqrt{x+a} + \sqrt{x+b}), \\ \quad \text{若 } x+a > 0 \text{ 及 } x+b > 0; \\ -2\ln(\sqrt{-x-a} + \sqrt{-x-b}), \\ \quad \text{若 } x+a < 0 \text{ 及 } x+b < 0. \end{cases}\end{aligned}$$

1790. $\int \sqrt{(x+a)(x+b)} dx.$

解 与上题相同,作同一代换,只是在求积分的过程中变动个别地方.今以 $x > -a$ 时为例,解法如下:

$$\begin{aligned}\int \sqrt{(x+a)(x+b)} dx &= 2(b-a)^2 \int \operatorname{sh}^2 t \operatorname{ch}^2 t dt \\ &= \frac{1}{2} (b-a)^2 \int \operatorname{sh}^2 2t dt \\ &= \frac{1}{4} (b-a)^2 \int (\operatorname{ch} 4t - 1) dt \\ &= \frac{1}{4} (b-a)^2 \left(\frac{1}{4} \operatorname{sh} 4t - t \right) + C_1 \\ &= \frac{1}{4} (b-a)^2 [\operatorname{sh} t \operatorname{ch} t (1 + 2\operatorname{sh}^2 t) - t] + C_1 \\ &= \frac{1}{4} (b-a)^2 \left[\sqrt{\frac{x+a}{b-a}} \cdot \sqrt{1 + \frac{x+a}{b-a}} \right.\end{aligned}$$

$$\text{解} \quad \int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x(\ln x - 1) + C.$$

$$1792. \quad \int x^n \ln x dx \quad (n \neq -1).$$

$$\begin{aligned} \text{解} \quad \int x^n \ln x dx &= \frac{1}{n+1} \int \ln x d(x^{n+1}) = \frac{1}{n+1} x^{n+1} \ln x \\ &\quad - \frac{1}{n+1} \int x^{n+1} \cdot \frac{1}{x} dx = \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C. \end{aligned}$$

$$1793. \quad \int \left(\frac{\ln x}{x} \right)^2 dx.$$

$$\begin{aligned} \text{解} \quad \int \left(\frac{\ln x}{x} \right)^2 dx &= - \int \ln^2 x d\left(\frac{1}{x} \right) \\ &= - \frac{1}{x} \ln^2 x + \int \frac{1}{x} \cdot 2 \ln x \cdot \frac{1}{x} dx \\ &= - \frac{1}{x} \ln^2 x - 2 \int \ln x d\left(\frac{1}{x} \right) = - \frac{1}{x} \ln^2 x - \frac{2}{x} \ln x \\ &\quad + 2 \int \frac{1}{x} \cdot \frac{1}{x} dx = - \frac{1}{x} (\ln^2 x + 2 \ln x + 2) + C. \end{aligned}$$

$$1794. \quad \int \sqrt{x} \ln^2 x dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{x} \ln^2 x dx &= \frac{2}{3} \int \ln^2 x d(x^{\frac{3}{2}}) \\ &= \frac{2}{3} x^{\frac{3}{2}} \ln^2 x - \frac{4}{3} \int x^{\frac{3}{2}} \ln x \cdot \frac{1}{x} dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \ln^2 x - \frac{8}{9} \int \ln x d(x^{\frac{3}{2}}) \\ &= \frac{2}{3} x^{\frac{3}{2}} \ln^2 x - \frac{8}{9} x^{\frac{3}{2}} \ln x + \frac{8}{9} \int x^{\frac{3}{2}} \cdot \frac{1}{x} dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \left(\ln^2 x - \frac{4}{3} \ln x + \frac{8}{9} \right) + C. \end{aligned}$$

$$1795. \quad \int x e^{-x} dx.$$

$$\begin{aligned}\text{解} \quad \int x e^{-x} dx &= - \int x d(e^{-x}) = - x e^{-x} + \int e^{-x} dx \\ &= - e^{-x}(x+1) + C.\end{aligned}$$

$$1796. \int x^2 e^{-2x} dx.$$

$$\begin{aligned}\text{解} \quad \int x^2 e^{-2x} dx &= - \frac{1}{2} \int x^2 d(e^{-2x}) \\ &= - \frac{1}{2} x^2 e^{-2x} + \frac{1}{2} \int e^{-2x} \cdot 2x dx \\ &= - \frac{1}{2} x^2 e^{-2x} - \frac{1}{2} \int x d(e^{-2x}) \\ &= - \frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx \\ &= - \frac{1}{2} e^{-2x} \left(x^2 + x + \frac{1}{2} \right) + C.\end{aligned}$$

$$1797. \int x^3 e^{-x^2} dx.$$

$$\begin{aligned}\text{解} \quad \int x^3 e^{-x^2} dx &= - \frac{1}{2} \int x^2 d(e^{-x^2}) \\ &= - \frac{1}{2} x^2 e^{-x^2} + \frac{1}{2} \int e^{-x^2} d(x^2) = - \frac{x^2 + 1}{2} e^{-x^2} + C.\end{aligned}$$

$$1798. \int x \cos x dx.$$

$$\begin{aligned}\text{解} \quad \int x \cos x dx &= \int x d(\sin x) \\ &= x \sin x - \int \sin x dx = x \sin x + \cos x + C.\end{aligned}$$

$$1799. \int x^2 \sin 2x dx.$$

$$\begin{aligned}\text{解} \quad \int x^2 \sin 2x dx &= - \frac{1}{2} \int x^2 d(\cos 2x) \\ &= - \frac{1}{2} x^2 \cos 2x + \frac{1}{2} \int x d(\sin 2x)\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int\sin 2x dx \\
&= -\frac{2x^2-1}{4}\cos 2x + \frac{1}{2}x\sin 2x + C.
\end{aligned}$$

1800. $\int x\operatorname{sh} x dx.$

解 $\int x\operatorname{sh} x dx = \int x d(\operatorname{ch} x)$
 $= x\operatorname{ch} x - \int \operatorname{ch} x dx = x\operatorname{ch} x - \operatorname{sh} x + C.$

1801. $\int x^3\operatorname{ch} 3x dx.$

解 $\int x^3\operatorname{ch} 3x dx = \frac{1}{3}\int x^3 d(\operatorname{sh} 3x)$
 $= \frac{1}{3}x^3\operatorname{sh} 3x - \int x^2\operatorname{sh} 3x dx$
 $= \frac{1}{3}x^3\operatorname{sh} 3x - \frac{1}{3}\int x^2 d(\operatorname{ch} 3x)$
 $= \frac{1}{3}x^3\operatorname{sh} 3x - \frac{1}{3}x^2\operatorname{ch} 3x + \frac{2}{3}\int x\operatorname{ch} 3x dx$
 $= \frac{1}{3}x^3\operatorname{sh} 3x - \frac{1}{3}x^2\operatorname{ch} 3x + \frac{2}{9}\int x d(\operatorname{sh} 3x)$
 $= \frac{1}{3}x^3\operatorname{sh} 3x - \frac{1}{3}x^2\operatorname{ch} 3x + \frac{2}{9}x\operatorname{sh} 3x - \frac{2}{9}\int \operatorname{sh} 3x dx$
 $= \left(\frac{x^3}{3} + \frac{2x}{9}\right)\operatorname{sh} 3x - \left(\frac{x^2}{3} + \frac{2}{27}\right)\operatorname{ch} 3x + C.$

1802. $\int \operatorname{arc} \operatorname{tg} x dx.$

解 $\int \operatorname{arc} \operatorname{tg} x dx = x\operatorname{arc} \operatorname{tg} x - \int \frac{x}{1+x^2} dx$
 $= x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C.$

1803. $\int \operatorname{arc} \sin x dx.$

$$\begin{aligned}\text{解 } \int \arcsin x dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \arcsin x + \sqrt{1-x^2} + C.\end{aligned}$$

$$1804. \int x \arctan x dx.$$

$$\begin{aligned}\text{解 } \int x \arctan x dx &= \frac{1}{2} \int \arctan x d(x^2) \\ &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{1+x^2}{2} \arctan x - \frac{x}{2} + C.\end{aligned}$$

$$1805. \int x^2 \arccos x dx.$$

$$\begin{aligned}\text{解 } \int x^2 \arccos x dx &= \frac{1}{3} \int \arccos x d(x^3) \\ &= \frac{1}{3} x^3 \arccos x + \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx \\ &= \frac{1}{3} x^3 \arccos x - \frac{1}{6} \int \frac{x^2}{\sqrt{1-x^2}} d(1-x^2) \\ &= \frac{1}{3} x^3 \arccos x - \frac{1}{6} \int \left(\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2} \right) \\ &\quad d(1-x^2) \\ &= \frac{1}{3} x^3 \arccos x - \frac{1}{3} \sqrt{1-x^2} + \frac{1}{9} (1-x^2)^{\frac{3}{2}} + C \\ &= \frac{1}{3} x^3 \arccos x - \frac{x^2+2}{9} \sqrt{1-x^2} + C.\end{aligned}$$

$$1806. \int \frac{\arcsin x}{x^2} dx.$$

$$\begin{aligned}\text{解} \quad \int \frac{\arcsin x}{x^2} dx &= - \int \arcsin x d\left(\frac{1}{x}\right) \\ &= -\frac{1}{x} \arcsin x + \int \frac{dx}{x \sqrt{1-x^2}}.\end{aligned}$$

作代换 $x = \sin t$, 得

$$\begin{aligned}\int \frac{dx}{x \sqrt{1-x^2}} &= \int \frac{\cos t \, dt}{\sin t \cos t} = \int \frac{dt}{\sin t} \\ &= \ln \left| \operatorname{tg} \frac{t}{2} \right|^{**} + C \\ &= \ln \left| \frac{\sin t}{1 + \cos t} \right| + C = - \ln \left| \frac{1 + \cos t}{\sin t} \right| + C \\ &= - \ln \left| \frac{1 + \sqrt{1-x^2}}{x} \right| + C,\end{aligned}$$

最后得

$$\begin{aligned}\int \frac{\arcsin x}{x^2} dx &= -\frac{1}{x} \arcsin x \\ &\quad - \ln \left| \frac{1 + \sqrt{1-x^2}}{x} \right| + C.\end{aligned}$$

*) 利用 1703 题的结果.

$$1807. \int \ln(x + \sqrt{1+x^2}) dx.$$

$$\begin{aligned}\text{解} \quad \int \ln(x + \sqrt{1+x^2}) dx \\ &= x \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx \\ &= x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.\end{aligned}$$

$$1808. \int x \ln \frac{1+x}{1-x} dx$$

$$\text{解} \quad \int x \ln \frac{1+x}{1-x} dx = \frac{1}{2} \int \ln \frac{1+x}{1-x} d(x^2)$$

$$\begin{aligned}
&= \frac{x^2}{2} \ln \frac{1+x}{1-x} - \int \frac{x^2}{1-x^2} dx \\
&= \frac{x^2}{2} \ln \frac{1+x}{1-x} + \int \left(1 - \frac{1}{1-x^2} \right) dx \\
&= x - \frac{1-x^2}{2} \ln \frac{1+x}{1-x} + C.
\end{aligned}$$

1809. $\int \operatorname{arc} \operatorname{tg} \sqrt{x} dx.$

解
$$\begin{aligned}
\int \operatorname{arc} \operatorname{tg} \sqrt{x} dx &= x \operatorname{arc} \operatorname{tg} \sqrt{x} \\
&\quad - \frac{1}{2} \int \frac{x}{\sqrt{x}(1+x)} dx \\
&= x \operatorname{arc} \operatorname{tg} \sqrt{x} - \int \left(1 - \frac{1}{1+x} \right) d(\sqrt{x}) \\
&= (x+1) \operatorname{arc} \operatorname{tg} \sqrt{x} - \sqrt{x} + C.
\end{aligned}$$

1810. $\int \sin x \cdot \ln(\operatorname{tg} x) dx.$

解
$$\begin{aligned}
\int \sin x \cdot \ln(\operatorname{tg} x) dx &= - \int \ln(\operatorname{tg} x) d(\cos x) \\
&= - \cos x \cdot \ln(\operatorname{tg} x) + \int \cos x \cdot \operatorname{ctg} x \cdot \sec^2 x dx \\
&= - \cos x \cdot \ln(\operatorname{tg} x) + \int \frac{dx}{\sin x} \\
&= - \cos x \cdot \ln(\operatorname{tg} x) + \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.
\end{aligned}$$

求下列积分：

1811. $\int x^5 e^{x^3} dx.$

解
$$\begin{aligned}
\int x^5 e^{x^3} dx &= \frac{1}{3} \int x^3 d(e^{x^3}) \\
&= \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} \int e^{x^3} d(x^3)
\end{aligned}$$

$$= \frac{1}{3}(x^3 - 1)e^{x^3} + C.$$

1812. $\int (\arcsin x)^2 dx.$

解
$$\begin{aligned}\int (\arcsin x)^2 dx &= x(\arcsin x)^2 - 2 \int \frac{x \arcsin x}{\sqrt{1-x^2}} dx \\ &= x(\arcsin x)^2 + 2 \int \arcsin x d(\sqrt{1-x^2}) \\ &= x(\arcsin x)^2 + 2 \sqrt{1-x^2} \arcsin x - 2 \int dx \\ &= x(\arcsin x)^2 + 2 \sqrt{1-x^2} \arcsin x - 2x + C.\end{aligned}$$

1813. $\int x(\arctg x)^2 dx.$

解
$$\begin{aligned}\int x(\arctg x)^2 dx &= \frac{1}{2} \int (\arctg x)^2 d(x^2) \\ &= \frac{1}{2} x^2 (\arctg x)^2 - \int \frac{x^2 \arctg x}{1+x^2} dx \\ &= \frac{x^2}{2} (\arctg x)^2 - \int \left(1 - \frac{1}{1+x^2} \right) \arctg x dx \\ &= \frac{x^2}{2} (\arctg x)^2 - \int \arctg x dx \\ &\quad + \int \arctg x d(\arctg x) \\ &= \frac{x^2}{2} (\arctg x)^2 - x \arctg x + \int \frac{x dx}{1+x^2} \\ &\quad + \frac{1}{2} (\arctg x)^2 \\ &= \frac{x^2+1}{2} (\arctg x)^2 - x \arctg x + \frac{1}{2} \ln(1+x^2) \\ &\quad + C.\end{aligned}$$

1814. $\int x^2 \ln \frac{1-x}{1+x} dx.$

$$\begin{aligned}
& \text{解} \quad \int x^2 \ln \frac{1-x}{1+x} dx \\
&= \frac{1}{3} \int \ln \frac{1-x}{1+x} d(x^3) \\
&= \frac{1}{3} x^3 \ln \frac{1-x}{1+x} + \frac{2}{3} \int \frac{x^3}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1-x}{1+x} + \frac{2}{3} \int \left(-x + \frac{x}{1-x^2} \right) dx \\
&= \frac{x^3}{3} \ln \frac{1-x}{1+x} - \frac{1}{3} x^2 - \frac{1}{3} \ln(1-x^2) + C.
\end{aligned}$$

$$1815. \int \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx.$$

$$\begin{aligned}
& \text{解} \quad \int \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx \\
&= \int \ln(x + \sqrt{1+x^2}) d(\sqrt{1+x^2}) \\
&= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) \\
&\quad - \int \sqrt{1+x^2} \cdot \frac{1}{\sqrt{1+x^2}} dx \\
&= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) - x + C.
\end{aligned}$$

$$1816. \int \frac{x^2}{(1+x^2)^2} dx.$$

$$\begin{aligned}
& \text{解} \quad \int \frac{x^2}{(1+x^2)^2} dx = \frac{1}{2} \int \frac{x}{(1+x^2)^2} d(1+x^2) \\
&= -\frac{1}{2} \int x d\left(\frac{1}{1+x^2}\right) = -\frac{x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{1+x^2} \\
&= -\frac{x}{2(1+x^2)} + \frac{1}{2} \arctan x + C.
\end{aligned}$$

$$1817. \int \frac{dx}{(a^2+x^2)^2}.$$

解 当 $a = 0$ 时,

$$\int \frac{dx}{(a^2 + x^2)^2} = \int \frac{dx}{x^4} = -\frac{1}{3x^3} + C,$$

当 $a \neq 0$ 时,

$$\begin{aligned} \int \frac{dx}{(a^2 + x^2)^2} &= \frac{1}{a^2} \int \frac{(a^2 + x^2) - x^2}{(a^2 + x^2)^2} dx \\ &= \frac{1}{a^2} \int \frac{dx}{a^2 + x^2} - \frac{1}{a^2} \int \frac{x^2}{(a^2 + x^2)^2} dx \\ &= \frac{1}{a^3} \operatorname{arc} \operatorname{tg} \frac{x}{a} - \frac{1}{a^3} \int \frac{\left(\frac{x}{a}\right)^2 d\left(\frac{x}{a}\right)}{\left[1 + \left(\frac{x}{a}\right)^2\right]^2} \\ &= \frac{1}{a^3} \operatorname{arc} \operatorname{tg} \frac{x}{a} \\ &\quad - \frac{1}{a^3} \left[-\frac{\frac{x}{a}}{2\left(1 + \frac{x^2}{a^2}\right)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} \frac{x}{a} \right]^{**} + C \\ &= \frac{1}{2a^3} \operatorname{arc} \operatorname{tg} \frac{x}{a} + \frac{x}{2a^2(a^2 + x^2)} + C. \end{aligned}$$

*) 利用 1816 题的结果.

1818. $\int \sqrt{a^2 - x^2} dx.$

$$\begin{aligned} \text{解 } \int \sqrt{a^2 - x^2} dx &= x \sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}} dx \\ &= x \sqrt{a^2 - x^2} + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}} dx \\ &= x \sqrt{a^2 - x^2} + a^2 \operatorname{arc} \sin \frac{x}{|a|} \\ &\quad - \int \sqrt{a^2 - x^2} dx, \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{4} \left(\frac{1}{3} x(a^2 + x^2) \sqrt{a^2 + x^2} - \frac{a^2}{3} \int \sqrt{a^2 + x^2} dx \right) \\
&= \frac{1}{4} x(a^2 + x^2) \sqrt{a^2 + x^2} \\
&\quad - \frac{a^2}{4} \left(\frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) \right) + C \\
&= \frac{x(2x^2 + a^2)}{8} \sqrt{a^2 + x^2} \\
&\quad - \frac{a^4}{8} \ln(x + \sqrt{x^2 + a^2}) + C.
\end{aligned}$$

*) 利用 1786 题的结果.

1821. $\int x \sin^2 x dx.$

$$\begin{aligned}
\text{解} \quad \int x \sin^2 x dx &= \frac{1}{2} \int x(1 - \cos 2x) dx \\
&= \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx \\
&= \frac{1}{4} x^2 - \frac{1}{4} \int x d(\sin 2x) \\
&= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x + \frac{1}{4} \int \sin 2x dx \\
&= \frac{1}{4} x^2 - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C.
\end{aligned}$$

1822. $\int e^{\sqrt{x}} dx.$

解 设 $\sqrt{x} = t$, 则 $x = t^2, dx = 2t dt$, 代入得

$$\begin{aligned}
\int e^{\sqrt{x}} dx &= 2 \int t e^t dt = 2 \int t d(e^t) \\
&= 2 t e^t - 2 \int e^t dt \\
&= 2 t e^t - 2 e^t + C = 2(\sqrt{x} - 1) e^{\sqrt{x}} + C.
\end{aligned}$$

1823. $\int x \sin \sqrt{x} dx.$

解 设 $\sqrt{x} = t$, 则 $x = t^2, dx = 2t dt$, 代入得

$$\begin{aligned} \int x \sin \sqrt{x} dx &= 2 \int t^3 \sin t dt = -2 \int t^3 d(\cos t) \\ &= -2t^3 \cos t + 6 \int t^2 \cos t dt \\ &= -2t^3 \cos t + 6 \int t^2 d(\sin t) \\ &= -2t^3 \cos t + 6t^2 \sin t - 12 \int t \sin t dt \\ &= -2t^3 \cos t + 6t^2 \sin t + 12 \int t d(\cos t) \\ &= -2t^3 \cos t + 6t^2 \sin t + 12t \cos t - 12 \int \cos t dt \\ &= -2(t^2 - 6)t \cos t + 6(t^2 - 2) \sin t + C \\ &= 2(6 - x) \sqrt{x} \cos \sqrt{x} - 6(2 - x) \sin \sqrt{x} + C. \end{aligned}$$

1824. $\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx.$

解
$$\begin{aligned} \int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx &= \int \frac{x}{\sqrt{1+x^2}} d(e^{\operatorname{arctg} x}) \\ &= \frac{x}{\sqrt{1+x^2}} e^{\operatorname{arctg} x} - \int e^{\operatorname{arctg} x} \cdot \frac{1}{(1+x^2)^{\frac{3}{2}}} dx \\ &= \frac{x}{\sqrt{1+x^2}} e^{\operatorname{arctg} x} - \int \frac{1}{\sqrt{1+x^2}} d(e^{\operatorname{arctg} x}) \\ &= \frac{x-1}{\sqrt{1+x^2}} e^{\operatorname{arctg} x} - \int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx, \end{aligned}$$

于是得

$$\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx = \frac{x-1}{2\sqrt{1+x^2}} e^{\operatorname{arctg} x} + C.$$

$$1825. \int \frac{e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx &= \int \frac{1}{\sqrt{1+x^2}} d(e^{\operatorname{arctg} x}) \\ &= \frac{1}{\sqrt{1+x^2}} e^{\operatorname{arctg} x} + \int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} dx \\ &= \frac{1}{\sqrt{1+x^2}} e^{\operatorname{arctg} x} + \frac{x-1}{2\sqrt{1+x^2}} e^{\operatorname{arctg} x} + C \\ &= \frac{x+1}{2\sqrt{1+x^2}} e^{\operatorname{arctg} x} + C. \end{aligned}$$

*) 利用 1824 题的结果.

$$1826. \int \sin(\ln x) dx.$$

$$\begin{aligned} \text{解} \quad \int \sin(\ln x) dx &= x \sin(\ln x) - \int x \cos(\ln x) \cdot \frac{1}{x} dx \\ &= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx, \end{aligned}$$

于是得

$$\int \sin(\ln x) dx = \frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C.$$

$$1827. \int \cos(\ln x) dx.$$

$$\begin{aligned} \text{解} \quad \int \cos(\ln x) dx &= x \cos(\ln x) + \int \sin(\ln x) dx \\ &= x \cos(\ln x) + \frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C \\ &= \frac{x}{2} [\sin(\ln x) + \cos(\ln x)] + C. \end{aligned}$$

*) 利用 1826 题的结果.

$$1828. \int e^{ax} \cos bx dx.$$

解 如果 a, b 同时为零, 积分显然为 $x + C$; 若 $a = 0$, $b \neq 0$, 积分显然为 $\frac{1}{b} \sin bx + C$; 以下设 $a \neq 0$:

$$\begin{aligned} \int e^{ax} \cos bx dx &= \frac{1}{a} \int \cos bx d(e^{ax}) \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx dx \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \int \sin bx d(e^{ax}) \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx dx \end{aligned}$$

于是得

$$\begin{aligned} \int e^{ax} \cos bx dx &= \frac{a^2}{a^2 + b^2} \left(\frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx \right) \\ &+ C = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} + C. \end{aligned}$$

1829. $\int e^{ax} \sin bx dx.$

解 若 $a = b = 0$, 则积分为 $x + C$; 以下设 $a^2 + b^2 \neq 0$, 则有

$$\begin{aligned} \int e^{ax} \sin bx dx &= \frac{1}{a} \int \sin bx d(e^{ax}) \\ &= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx dx \\ &= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} \int \cos bx d(e^{ax}) \\ &= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx dx, \end{aligned}$$

故 $\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} + C.$

1830. $\int e^{2x} \sin^2 x dx.$

$$\begin{aligned}
\text{解} \quad \int e^{2x} \sin^2 x dx &= \frac{1}{2} \int e^{2x} (1 - \cos 2x) dx \\
&= \frac{1}{2} \int e^{2x} dx - \frac{1}{2} \int e^{2x} \cos 2x dx \\
&= \frac{1}{4} e^{2x} - \frac{1}{2} \left(\frac{2 \cos 2x + 2 \sin 2x}{8} e^{2x} \right)^{**} + C \\
&= \frac{1}{8} e^{2x} (2 - \cos 2x - \sin 2x) + C.
\end{aligned}$$

*) 利用 1828 题的结果.

$$1831. \int (e^x - \cos x)^2 dx.$$

$$\begin{aligned}
\text{解} \quad \int (e^x - \cos x)^2 dx &= \int (e^{2x} - 2e^x \cos x + \cos^2 x) dx \\
&= \frac{1}{2} e^{2x} - 2 \cdot \frac{e^x (\cos x + \sin x)^{**}}{2} + \\
&\quad \left(\frac{x}{2} + \frac{1}{4} \sin 2x \right)^{***} + C \\
&= \frac{1}{2} e^{2x} - e^x (\cos x + \sin x) + \frac{x}{2} + \frac{1}{4} \sin 2x + C.
\end{aligned}$$

*) 利用 1828 题的结果.

**) 利用 1742 题的结果.

$$1832. \int \frac{\operatorname{arc} \operatorname{ctg} e^x}{e^x} dx.$$

$$\begin{aligned}
\text{解} \quad \int \frac{\operatorname{arc} \operatorname{ctg} e^x}{e^x} dx &= - \int \operatorname{arc} \operatorname{ctg} e^x d(e^{-x}) \\
&= - e^{-x} \cdot \operatorname{arc} \operatorname{ctg}(e^x) - \int \frac{dx}{1 + e^{2x}} \\
&= - e^{-x} \operatorname{arc} \operatorname{ctg}(e^x) - \frac{1}{2} [2x - \ln(1 + e^{2x})]^{**} + C. \\
&= - e^{-x} \operatorname{arc} \operatorname{ctg}(e^x) - x + \frac{1}{2} \ln(1 + e^{2x}) + C.
\end{aligned}$$

*) 利用 1759 题的结果.

$$1833. \int \frac{\ln(\sin x)}{\sin^2 x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\ln(\sin x)}{\sin^2 x} dx &= - \int \ln(\sin x) d(\operatorname{ctg} x) \\ &= - \operatorname{ctg} x \cdot \ln(\sin x) + \int \operatorname{ctg}^2 x dx \\ &= - \operatorname{ctg} x \cdot \ln(\sin x) + (- \operatorname{ctg} x - x)^{*)} + C \\ &= - [x + \operatorname{ctg} x \cdot \ln(\operatorname{esin} x)] + C. \end{aligned}$$

*) 利用 1649 题或 1751 题的结果.

$$1834. \int \frac{x dx}{\cos^2 x}$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{\cos^2 x} &= \int x d(\operatorname{tg} x) = x \operatorname{tg} x - \int \operatorname{tg} x dx \\ &= x \operatorname{tg} x + \ln |\cos x|^{*)} + C. \end{aligned}$$

*) 利用 1697 题的结果.

$$1835. \int \frac{x e^x}{(x+1)^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x e^x}{(x+1)^2} dx &= - \int x e^x d\left(\frac{1}{1+x}\right) \\ &= - \frac{x}{1+x} e^x + \int - \frac{1}{1+x} e^x (x+1) dx \\ &= - \frac{x}{1+x} e^x + e^x + C = \frac{e^x}{1+x} + C. \end{aligned}$$

下列积分的求法需要把二次三项式化成正则型, 并利用下列公式:

$$\text{I. } \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arc} \operatorname{tg} \frac{x}{a} + C \quad (a \neq 0);$$

$$\text{II. } \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad (a \neq 0);$$

$$\text{III. } \int \frac{x dx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln |a^2 \pm x^2| + C;$$

$$\text{IV. } \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0);$$

$$\text{V. } \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C;$$

$$\text{VI. } \int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2} + C;$$

$$\begin{aligned} \text{VII. } \int \sqrt{a^2 - x^2} dx \\ = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C \quad (a > 0); \end{aligned}$$

$$\begin{aligned} \text{VIII. } \int \sqrt{x^2 \pm a^2} dx \\ = \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln |x + \sqrt{x^2 \pm a^2}| + C. \end{aligned}$$

求下列积分:

$$1836^+. \int \frac{dx}{a + bx^2} \quad (ab \neq 0)$$

解 当 $ab > 0$ 时,

$$\begin{aligned} & \int \frac{dx}{a + bx^2} \\ &= \operatorname{sgn} a \cdot \frac{1}{\sqrt{|b|}} \int \frac{d(\sqrt{|b|x})}{(\sqrt{|a|})^2 + (\sqrt{|b|x})^2} \\ &= \operatorname{sgn} a \cdot \frac{1}{\sqrt{ab}} \operatorname{arctg} \left(x \sqrt{\frac{b}{a}} \right) + C; \end{aligned}$$

当 $ab < 0$ 时,

$$\begin{aligned} & \int \frac{dx}{a + bx^2} = \operatorname{sgn} a \cdot \int \frac{dx}{|a| - |b|x^2} \\ &= \operatorname{sgn} a \cdot \frac{1}{\sqrt{|b|}} \int \frac{d(\sqrt{|b|x})}{(\sqrt{|a|})^2 - (\sqrt{|b|x})^2} \end{aligned}$$

$$= \frac{\operatorname{sgn} a}{2 \sqrt{-ab}} \ln \left| \frac{\sqrt{|a|} + x \sqrt{|b|}}{\sqrt{|a|} - x \sqrt{|b|}} \right| + C.$$

$$1837. \int \frac{dx}{x^2 - x + 2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x^2 - x + 2} &= \int \frac{d\left(x - \frac{1}{2}\right)}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\ &= \frac{2}{\sqrt{7}} \operatorname{arc} \operatorname{tg} \frac{2x - 1}{\sqrt{7}} + C. \end{aligned}$$

$$1838. \int \frac{dx}{3x^2 - 2x - 1}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{3x^2 - 2x - 1} &= \frac{1}{3} \int \frac{dx}{x^2 - \frac{2}{3}x - \frac{1}{3}} \\ &= \frac{1}{3} \int \frac{d\left(x - \frac{1}{3}\right)}{\left(x - \frac{1}{3}\right)^2 - \left(\frac{2}{3}\right)^2} \\ &= -\frac{1}{3} \cdot \frac{3}{4} \ln \left| \frac{\frac{2}{3} + \left(x - \frac{1}{3}\right)}{\frac{2}{3} - \left(x - \frac{1}{3}\right)} \right| + C_1 \\ &= \frac{1}{4} \ln \left| \frac{x - 1}{3x + 1} \right| + C. \end{aligned}$$

$$1839. \int \frac{x dx}{x^4 - 2x^2 - 1}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{x^4 - 2x^2 - 1} &= \frac{1}{2} \int \frac{d(x^2 - 1)}{(x^2 - 1)^2 - (\sqrt{2})^2} \\ &= \frac{1}{4 \sqrt{2}} \ln \left| \frac{x^2 - (\sqrt{2} + 1)}{x^2 + (\sqrt{2} - 1)} \right| + C. \end{aligned}$$

$$1840. \int \frac{x+1}{x^2+x+1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x+1}{x^2+x+1} dx &= \int \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1} dx \\ &= \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1}{2} \ln(x^2+x+1) + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg}\left(\frac{2x+1}{\sqrt{3}}\right) + C. \end{aligned}$$

$$1841. \int \frac{x dx}{x^2 - 2x \cos \alpha + 1}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{x^2 - 2x \cos \alpha + 1} &= \int \frac{x - \cos \alpha + \cos \alpha}{(x - \cos \alpha)^2 + \sin^2 \alpha} dx \\ &= \frac{1}{2} \int \frac{d[(x - \cos \alpha)^2 + \sin^2 \alpha]}{(x - \cos \alpha)^2 + \sin^2 \alpha} \\ &\quad + \cos \alpha \cdot \int \frac{d(x - \cos \alpha)}{(x - \cos \alpha)^2 + \sin^2 \alpha} \\ &= \frac{1}{2} \ln(x^2 - 2x \cos \alpha + 1) \\ &\quad + \operatorname{ctg} \alpha \cdot \operatorname{arctg}\left(\frac{x - \cos \alpha}{\sin \alpha}\right) + C \\ &\quad (a \neq k\pi, k = 0, \pm 1, \pm 2, \dots). \end{aligned}$$

$$1842. \int \frac{x^3 dx}{x^4 - x^2 + 2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3 dx}{x^4 - x^2 + 2} \\ &= \frac{1}{2} \int \frac{x^2 d(x^2)}{\left(x^2 - \frac{1}{2}\right)^2 + \frac{7}{4}} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \ln |x^6 - x^3 - 2| \\
&\quad - \frac{1}{18} \ln \left| \frac{\frac{3}{2} + \left(x^3 - \frac{1}{2}\right)}{\frac{3}{2} - \left(x^3 - \frac{1}{2}\right)} \right| + C \\
&= \frac{1}{9} \ln \{ |x^3 + 1| \cdot (x^3 - 2)^2 \} + C.
\end{aligned}$$

如果本题不化成正则型来作, 则有更简单的作法, 事实上,

$$\begin{aligned}
\int \frac{x^5 dx}{x^6 - x^3 - 2} &= \frac{1}{3} \int \frac{x^3 d(x^3)}{(x^3 - 2)(x^3 + 1)} \\
&= \frac{1}{9} \int \left(\frac{2}{x^3 - 2} + \frac{1}{x^3 + 1} \right) d(x^3) \\
&= \frac{1}{9} \ln \{ |x^3 + 1| \cdot (x^3 - 2)^2 \} + C.
\end{aligned}$$

1844. $\int \frac{dx}{3\sin^2 x - 8\sin x \cos x + 5\cos^2 x}.$

解
$$\begin{aligned}
&\int \frac{dx}{3\sin^2 x - 8\sin x \cos x + 5\cos^2 x} \\
&= \int \frac{d(\operatorname{tg} x)}{3\operatorname{tg}^2 x - 8\operatorname{tg} x + 5} \\
&= \frac{1}{3} \int \frac{d\left(\operatorname{tg} x - \frac{4}{3}\right)}{\left(\operatorname{tg} x - \frac{4}{3}\right)^2 - \left(\frac{1}{3}\right)^2} \\
&= \frac{1}{2} \ln \left| \frac{\frac{1}{3} - \left(\operatorname{tg} x - \frac{4}{3}\right)}{\frac{1}{3} + \left(\operatorname{tg} x - \frac{4}{3}\right)} \right| + C_1 \\
&= \frac{1}{2} \ln \left| \frac{3\sin x - 5\cos x}{\sin x - \cos x} \right| + C.
\end{aligned}$$

$$1845. \int \frac{dx}{\sin x + 2\cos x + 3}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin x + 2\cos x + 3} &= \int \frac{\frac{1}{\cos^2 \frac{x}{2}} dx}{2\operatorname{tg} \frac{x}{2} + 4 + \sec^2 \frac{x}{2}} \\ &= 2 \int \frac{d\left(\operatorname{tg} \frac{x}{2}\right)}{\left(\operatorname{tg} \frac{x}{2} + 1\right)^2 + 4} = \operatorname{arc} \operatorname{tg} \left[\frac{\operatorname{tg} \frac{x}{2} + 1}{2} \right] + C. \end{aligned}$$

$$1846. \int \frac{dx}{\sqrt{a + bx^2}} \quad (b \neq 0).$$

解 当 $b > 0$ 时,

$$\int \frac{dx}{\sqrt{a + bx^2}} = \frac{1}{\sqrt{b}} \ln |x \sqrt{b} + \sqrt{a + bx^2}| + C;$$

当 $a > 0$ 及 $b < 0$ 时,

$$\begin{aligned} \int \frac{dx}{\sqrt{a + bx^2}} &= \frac{1}{\sqrt{-b}} \int \frac{d(\sqrt{-bx})}{\sqrt{(\sqrt{a})^2 - (\sqrt{-bx})^2}} \\ &= \frac{1}{\sqrt{-b}} \operatorname{arc} \sin \left[x \sqrt{-\frac{b}{a}} \right] + C. \end{aligned}$$

$$1847. \int \frac{dx}{\sqrt{1 - 2x - x^2}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{1 - 2x - x^2}} &= \int \frac{d(x + 1)}{\sqrt{2 - (x + 1)^2}} \\ &= \operatorname{arc} \sin \left(\frac{x + 1}{\sqrt{2}} \right) + C. \end{aligned}$$

$$1848. \int \frac{dx}{\sqrt{x + x^2}}.$$

$$\begin{aligned}\text{解} \quad \int \frac{dx}{\sqrt{x+x^2}} &= \int \frac{d\left(x + \frac{1}{2}\right)}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}} \\ &= \ln \left| x + \frac{1}{2} + \sqrt{x+x^2} \right| + C.\end{aligned}$$

本题即 1687 题, 注意不同的解法及不同形式的结果.

$$1849. \int \frac{dx}{\sqrt{2x^2 - x + 2}}.$$

$$\begin{aligned}\text{解} \quad \int \frac{dx}{\sqrt{2x^2 - x + 2}} &= \frac{1}{\sqrt{2}} \int \frac{d\left(x - \frac{1}{4}\right)}{\sqrt{\left(x - \frac{1}{4}\right)^2 + \frac{15}{16}}} \\ &= \frac{1}{\sqrt{2}} \ln \left| x - \frac{1}{4} + \sqrt{x^2 - \frac{x}{2} + 1} \right| + C.\end{aligned}$$

1850. 证明: 若

$$y = ax^2 + bx + c (a \neq 0),$$

$$\text{则当 } a > 0 \text{ 时, } \int \frac{dx}{\sqrt{y}} = \frac{1}{\sqrt{a}} \ln \left| \frac{y'}{2} + \sqrt{ay} \right| + C;$$

$$\text{当 } a < 0 \text{ 时, } \int \frac{dx}{\sqrt{y}} = \frac{1}{\sqrt{-a}} \arcsin \frac{-y'}{\sqrt{b^2 - 4ac}} + C.$$

证 当 $a > 0$ 时,

$$\begin{aligned}\int \frac{dx}{\sqrt{y}} &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}} \\ &= \frac{1}{\sqrt{a}} \int \frac{d\left(x + \frac{b}{2a}\right)}{\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}}}\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\sqrt{a}} \ln \left| x + \frac{b}{2a} + \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} \right| + C \\
&= -\frac{1}{\sqrt{a}} \ln \left| \frac{y'}{2} + \sqrt{ay} \right| + C;
\end{aligned}$$

当 $a < 0$ 时,

$$\begin{aligned}
\int \frac{dx}{\sqrt{y}} &= \frac{1}{\sqrt{-a}} \int \frac{dx}{\sqrt{-x^2 - \frac{b}{a}x - \frac{c}{a}}} \\
&= \frac{1}{\sqrt{-a}} \int \frac{d\left(x + \frac{b}{2a}\right)}{\sqrt{\frac{b^2 - 4ac}{4a^2} - \left(x + \frac{b}{2a}\right)^2}} \\
&= \frac{1}{\sqrt{-a}} \arcsin \frac{x + \frac{b}{2a}}{\frac{\sqrt{b^2 - 4ac}}{-2a}} + C \\
&= \frac{1}{\sqrt{-a}} \arcsin \left(\frac{-y'}{\sqrt{b^2 - 4ac}} \right) + C.
\end{aligned}$$

1851. $\int \frac{x dx}{\sqrt{5+x-x^2}}.$

解
$$\begin{aligned}
\int \frac{x dx}{\sqrt{5+x-x^2}} &= \int \frac{\left(x - \frac{1}{2}\right) + \frac{1}{2}}{\sqrt{\frac{21}{4} - \left(x - \frac{1}{2}\right)^2}} dx \\
&= -\frac{1}{2} \int \frac{d\left(\frac{21}{4} - \left(x - \frac{1}{2}\right)^2\right)}{\sqrt{\frac{21}{4} - \left(x - \frac{1}{2}\right)^2}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int \frac{d\left(x - \frac{1}{2}\right)}{\sqrt{\frac{21}{4} - \left(x - \frac{1}{2}\right)^2}} \\
& = -\sqrt{5+x-x^2} + \frac{1}{2} \arcsin\left(\frac{2x-1}{\sqrt{21}}\right) + C.
\end{aligned}$$

1852. $\int \frac{x+1}{\sqrt{x^2+x+1}} dx.$

解
$$\begin{aligned}
\int \frac{x+1}{\sqrt{x^2+x+1}} dx &= \int \frac{\left(x + \frac{1}{2}\right) + \frac{1}{2}}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}} dx \\
&= \frac{1}{2} \int \frac{d\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}} + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\
&= \sqrt{x^2+x+1} + \frac{1}{2} \ln\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right) \\
&\quad + C.
\end{aligned}$$

1853. $\int \frac{x dx}{\sqrt{1-3x^2-2x^4}}.$

解
$$\begin{aligned}
& \int \frac{x dx}{\sqrt{1-3x^2-2x^4}} \\
&= \frac{1}{2\sqrt{2}} \int \frac{d\left(x^2 + \frac{3}{4}\right)}{\sqrt{\frac{17}{16} - \left(x^2 + \frac{3}{4}\right)^2}} \\
&= \frac{1}{2\sqrt{2}} \arcsin\left(\frac{4x^2+3}{\sqrt{17}}\right) + C.
\end{aligned}$$

$$1854. \int \frac{x^3 dx}{\sqrt{x^4 - 2x^2 - 1}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3 dx}{\sqrt{x^4 - 2x^2 - 1}} &= \frac{1}{2} \int \frac{x^2 d(x^2)}{\sqrt{(x^2 - 1)^2 - 2}} \\ &= \frac{1}{2} \int \frac{(x^2 - 1) d(x^2 - 1)}{\sqrt{(x^2 - 1)^2 - 2}} + \frac{1}{2} \int \frac{d(x^2 - 1)}{\sqrt{(x^2 - 1)^2 - 2}} \\ &= \frac{1}{2} \sqrt{x^4 - 2x^2 - 1} \\ &\quad + \frac{1}{2} \ln |x^2 - 1 + \sqrt{x^4 - 2x^2 - 1}| + C. \end{aligned}$$

$$1855. \int \frac{x + x^3}{\sqrt{1 + x^2 - x^4}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x + x^3}{\sqrt{1 + x^2 - x^4}} dx &= \frac{1}{2} \int \frac{(1 + x^2) d(x^2)}{\sqrt{\frac{5}{4} - \left(x^2 - \frac{1}{2}\right)^2}} \\ &= \frac{1}{2} \int \frac{\left(x^2 - \frac{1}{2}\right) d\left(x^2 - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x^2 - \frac{1}{2}\right)^2}} \\ &\quad + \frac{3}{4} \int \frac{d\left(x^2 - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x^2 - \frac{1}{2}\right)^2}} \\ &= -\frac{1}{2} \sqrt{1 + x^2 - x^4} + \frac{3}{4} \arcsin \left(\frac{2x^2 - 1}{\sqrt{5}} \right) + C. \end{aligned}$$

$$1856. \int \frac{dx}{x \sqrt{x^2 + x + 1}}.$$

$$\text{解} \quad \text{作代换 } t = \frac{1}{x}, \text{ 则}$$

$x \sqrt{x^2 + x + 1} = \frac{\operatorname{sgn} t}{t^2} \sqrt{t^2 + t + 1}, dx = -\frac{dt}{t^2}$. 于是

$$\begin{aligned} \int \frac{dx}{x \sqrt{x^2 + x + 1}} &= -(\operatorname{sgn} t) \int \frac{dt}{\sqrt{t^2 + t + 1}} \\ &= -(\operatorname{sgn} t) \ln \left| t + \frac{1}{2} + \sqrt{t^2 + t + 1} \right| + C_1 \\ &= -(\operatorname{sgn} x) \ln \left| \frac{x + 2 + 2(\operatorname{sgn} x) \sqrt{x^2 + x + 1}}{2x} \right| \\ &\quad + C_1. \end{aligned}$$

故当 $x > 0$ 时,

$$\begin{aligned} \int \frac{dx}{x \sqrt{x^2 + x + 1}} \\ = -\ln \left| \frac{x + 2 + 2 \sqrt{x^2 + x + 1}}{x} \right| + C. \end{aligned}$$

当 $x < 0$ 时,

$$\begin{aligned} \int \frac{dx}{x \sqrt{x^2 + x + 1}} \\ = -\ln \left| \frac{2x}{x + 2 - 2 \sqrt{x^2 + x + 1}} \right| + C_1 \\ = -\ln \left| \frac{2x(x + 2 + 2 \sqrt{x^2 + x + 1})}{(x + 2)^2 - 4(x^2 + x + 1)} \right| + C_1 \\ = -\ln \left| \frac{x + 2 + 2 \sqrt{x^2 + x + 1}}{x} \right| + C. \end{aligned}$$

总之, 不论 x 为正或为负, 均有

$$\begin{aligned} \int \frac{dx}{x \sqrt{x^2 + x + 1}} \\ = -\ln \left| \frac{x + 2 + 2 \sqrt{x^2 + x + 1}}{x} \right| + C. \end{aligned}$$

*) 利用 1850 题的结果.

$$1857. \int \frac{dx}{x^2 \sqrt{x^2 + x - 1}}.$$

解 作代换 $t = \frac{1}{x}$, 则 $x^2 \sqrt{x^2 + x - 1}$

$$= \operatorname{sgn} t \cdot \frac{\sqrt{-t^2 + t + 1}}{t^3}, dx = -\frac{dt}{t^2}. \text{ 于是}$$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + x - 1}} &= -(\operatorname{sgn} t) \int \frac{t}{\sqrt{-t^2 + t + 1}} dt \\ &= -(\operatorname{sgn} t) \cdot \left(-\frac{1}{2} \int \frac{d(-t^2 + t + 1)}{\sqrt{-t^2 + t + 1}} \right. \\ &\quad \left. + \frac{1}{2} \int \frac{dt}{\sqrt{-t^2 + t + 1}} \right) \\ &= -(\operatorname{sgn} t) \cdot \left(-\sqrt{-t^2 + t + 1} \right. \\ &\quad \left. + \frac{1}{2} \arcsin \frac{2t - 1}{\sqrt{5}} \right) + C \\ &= (\operatorname{sgn} x) \cdot \left[\frac{\sqrt{x^2 + x - 1}}{|x|} \right. \\ &\quad \left. + \frac{1}{2} \arcsin \left(\frac{x - 2}{x \sqrt{5}} \right) \right] + C \\ &= \frac{\sqrt{x^2 + x - 1}}{x} + \frac{1}{2} \arcsin \left(\frac{x - 2}{|x| \sqrt{5}} \right) + C. \end{aligned}$$

其存在域为 $\left| x + \frac{1}{2} \right| > \frac{\sqrt{5}}{2}$.

*) 利用 1850 题的结果.

$$1858. \int \frac{dx}{(x+1) \sqrt{x^2 + 1}}.$$

解 设 $y = x + 1$, 本题即转化为 1856 题的类型. 由于解法类似, 且 $x + 1$ 的符号对结果没有影响, 故仅就

$x + 1 > 0$ 列出解法的主要步骤如下:

$$\begin{aligned} \int \frac{dx}{(x+1)\sqrt{x^2+1}} &= \int \frac{dy}{y\sqrt{y^2-2y+2}} \\ &= - \int \frac{d\left(\frac{1}{y}\right)}{\sqrt{\frac{2}{y^2} - \frac{2}{y} + 1}} \\ &= - \frac{1}{\sqrt{2}} \ln \left| \frac{1}{y} - \frac{1}{2} + \frac{\sqrt{y^2-2y+2}}{y\sqrt{2}} \right| + C_1 \\ &= - \frac{1}{\sqrt{2}} \ln \left| \frac{1-x+\sqrt{2(x^2+1)}}{x+1} \right| + C. \end{aligned}$$

1859. $\int \frac{dx}{(x-1)\sqrt{x^2-2}}.$

解 设 $x-1 = \frac{1}{t}$, 则

$$(x-1)\sqrt{x^2-2} = \frac{\sqrt{1+2t-t^2}}{t|t|}, dx = -\frac{1}{t^2}dt,$$

代入得

$$\begin{aligned} \int \frac{dx}{(x-1)\sqrt{x^2-2}} &= - \int \frac{\operatorname{sgnt} dt}{\sqrt{1+2t-t^2}} \\ &= - \operatorname{sgnt} \cdot \arcsin\left(\frac{t-1}{\sqrt{2}}\right) + C \\ &= \arcsin\left(\frac{x-2}{|x-1|\sqrt{2}}\right) + C \quad (|x| > \sqrt{2}). \end{aligned}$$

1860+. $\int \frac{dx}{(x+2)^2\sqrt{x^2+2x-5}}.$

解 设 $x+2 = \frac{1}{t}$, 则

$$(x+2)^2\sqrt{x^2+2x-5} = \frac{\sqrt{1-2t-5t^2}}{t^2|t|},$$

$$\begin{aligned}
&\text{解} \quad \int \sqrt{2+x+x^2} dx \\
&= \int \sqrt{\frac{7}{4} + \left(x + \frac{1}{2}\right)^2} d\left(x + \frac{1}{2}\right) \\
&= \frac{2x+1}{4} \sqrt{2+x+x^2} \\
&\quad + \frac{7}{8} \ln\left(x + \frac{1}{2} + \sqrt{2+x+x^2}\right) + C.
\end{aligned}$$

$$1863. \quad \int \sqrt{x^4 + 2x^2 - 1} x dx.$$

$$\begin{aligned}
&\text{解} \quad \int \sqrt{x^4 + 2x^2 - 1} x dx \\
&= \frac{1}{2} \int \sqrt{(x^2 + 1)^2 - 2} d(x^2 + 1) \\
&= \frac{x^2 + 1}{4} \sqrt{x^4 + 2x^2 - 1} \\
&\quad - \frac{1}{2} \ln(x^2 + 1 + \sqrt{x^4 + 2x^2 - 1}) + C.
\end{aligned}$$

$$1864. \quad \int \frac{1-x+x^2}{x \sqrt{1+x-x^2}} dx.$$

解 由于

$$\begin{aligned}
&\int \frac{dx}{x \sqrt{1+x-x^2}} \\
&= -\ln \left| \frac{2+x+2\sqrt{1+x-x^2}}{x} \right| + C_1
\end{aligned}$$

(可仿照 1856 题求得),

$$\begin{aligned}
&\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{d\left(x - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2}} \\
&= \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C_2,
\end{aligned}$$

$$\begin{aligned}\int \frac{x dx}{\sqrt{1+x-x^2}} &= \int \frac{\left(x - \frac{1}{2}\right) + \frac{1}{2}}{\sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2}} d\left(x - \frac{1}{2}\right) \\ &= -\sqrt{1+x-x^2} + \frac{1}{2} \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C_3,\end{aligned}$$

所以

$$\begin{aligned}\int \frac{1-x+x^2}{x\sqrt{1+x-x^2}} dx &= \int \frac{dx}{x\sqrt{1+x-x^2}} \\ &\quad - \int \frac{dx}{\sqrt{1+x-x^2}} + \int \frac{x dx}{\sqrt{1+x-x^2}} \\ &= -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| \\ &\quad + \frac{1}{2} \arcsin\left(\frac{1-2x}{\sqrt{5}}\right) - \sqrt{1+x-x^2} + C,\end{aligned}$$

其中存在域为满足不等式 $1+x-x^2 > 0$ 且 $x \neq 0$ 的一切 x 值, 即 $\left|x - \frac{1}{2}\right| < \frac{\sqrt{5}}{2}$ 及 $x \neq 0$.

1865. $\int \frac{x^2+1}{x\sqrt{x^4+1}} dx.$

解 $\int \frac{x^2+1}{x\sqrt{x^4+1}} dx.$

$$\begin{aligned}&= \int \frac{\operatorname{sgn} x \cdot \left(1 + \frac{1}{x^2}\right)}{\sqrt{x^2 + \frac{1}{x^2}}} dx = \int \frac{\operatorname{sgn} x d\left(x - \frac{1}{x}\right)}{\sqrt{\left(x - \frac{1}{x}\right)^2 + 2}} \\ &= \operatorname{sgn} x \cdot \ln\left|x - \frac{1}{x} + \sqrt{\left(x - \frac{1}{x}\right)^2 + 2}\right| + C_1 \\ &= \ln\left|\frac{x^2-1+\sqrt{x^4+1}}{x}\right| + C.\end{aligned}$$

§ 2. 有理函数的积分法

利用待定系数法,求下列积分:

$$1866. \int \frac{2x+3}{(x-2)(x+5)} dx.$$

解 设 $\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$, 通分后应有

$$2x+3 \equiv A(x+5) + B(x-2).$$

在这恒等式中,

令 $x=2$, 得 $7=7A, A=1$;

令 $x=-5$, 得 $-7=-7B, B=1$.

于是,

$$\begin{aligned} \int \frac{2x+3}{(x-2)(x+5)} dx &= \int \left(\frac{1}{x-2} + \frac{1}{x+5} \right) dx \\ &= \ln |(x-2)(x+5)| + C. \end{aligned}$$

*) 注意,这是一种习惯的说法. 实际上,不能直接令 $x=2$ (因为上述恒等式是当 $x \neq 2, x \neq -5$ 时得出来的), 而应令 $x \rightarrow 2$ 取极限, 得 $7=7A$, 以下类似情况都作此理解.

$$1867. \int \frac{x dx}{(x+1)(x+2)(x+3)}.$$

解 设 $\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$,

通分后应有

$$x \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)$$

$$\cdot (x+2).$$

在这恒等式中,

$$\text{令 } x = -1, \text{ 得 } -1 = 2A, A = -\frac{1}{2};$$

$$\text{令 } x = -2, \text{ 得 } -2 = -B, B = 2;$$

$$\text{令 } x = -3, \text{ 得 } -3 = 2C, C = -\frac{3}{2}.$$

于是,

$$\begin{aligned} & \int \frac{x dx}{(x+1)(x+2)(x+3)} \\ &= \int \left[\frac{-\frac{1}{2}}{x+1} + \frac{2}{x+2} + \frac{-\frac{3}{2}}{x+3} \right] dx \\ &= -\frac{1}{2} \ln|x+1| + 2 \ln|x+2| - \frac{3}{2} \ln|x+3| \\ & \quad + C \\ &= \frac{1}{2} \ln \left| \frac{(x+2)^4}{(x+1)(x+3)^3} \right| + C. \end{aligned}$$

$$1868. \int \frac{x^{10}}{x^2+x-2} dx.$$

$$\text{解} \quad \frac{x^{10}}{x^2+x-2} = x^8 - x^7 + 3x^6 - 5x^5 + 11x^4$$

$$- 21x^3 + 43x^2 - 85x + 171 + \frac{-341x + 342}{x^2+x-2},$$

$$\text{设 } \frac{-341x + 342}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}, \text{ 通分后应有}$$

$$-341x + 342 \equiv A(x-1) + B(x+2).$$

在这恒等式中,

$$\text{令 } x = -2, \text{ 得 } 1024 = -3A, A = -\frac{1024}{3};$$

令 $x = 1$, 得 $1 = 3B, B = \frac{1}{3}$.

于是,

$$\begin{aligned} \int \frac{x^{10}}{x^2 + x - 2} dx &= \int \left(x^8 - x^7 + 3x^6 - 5x^5 + 11x^4 \right. \\ &\quad \left. - 21x^3 + 43x^2 - 85x + 171 \right. \\ &\quad \left. - \frac{1024}{3(x+2)} + \frac{1}{3(x-1)} \right) dx \\ &= \frac{x^9}{9} - \frac{x^8}{8} + \frac{3x^7}{7} - \frac{5x^6}{6} + \frac{11x^5}{5} - \frac{21x^4}{4} + \frac{43x^3}{3} - \frac{85x^2}{2} \\ &\quad + 171x + \frac{1}{3} \ln \left| \frac{x-1}{(x+2)^{1024}} \right| + C. \end{aligned}$$

1869. $\int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx.$

解 $\frac{x^3 + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x}$
 $= 1 + \frac{5x^2 - 6x + 1}{x(x-2)(x-3)},$

设 $\frac{5x^2 - 6x + 1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$, 通分
 后应有

$$5x^2 - 6x + 1 \equiv A(x-2)(x-3) + Bx(x-3) + Cx(x-2).$$

在这恒等式中,

令 $x = 0$, 得 $1 = 6A, A = \frac{1}{6},$

令 $x = 2$, 得 $9 = -2B, B = -\frac{9}{2},$

令 $x = 3$, 得 $28 = 3C, C = \frac{28}{3}.$

于是,

$$\begin{aligned}
& \int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx \\
&= \int \left[1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)} \right] dx \\
&= x + \frac{1}{6} \ln|x| - \frac{9}{2} \ln|x-2| + \frac{28}{3} \ln|x-3| \\
&\quad + C.
\end{aligned}$$

1870. $\int \frac{x^4}{x^4 + 5x^2 + 4} dx.$

解 $\frac{x^4}{x^4 + 5x^2 + 4} = 1 + \frac{-(5x^2 + 4)}{(x^2 + 1)(x^2 + 4)}.$

设 $\frac{-(5x^2 + 4)}{(x^2 + 1)(x^2 + 4)} = \frac{A_1x + B_1}{x^2 + 1} + \frac{A_2x + B_2}{x^2 + 4}.$

通分后应有

$$\begin{aligned}
-(5x^2 + 4) &\equiv (A_1x + B_1)(x^2 + 4) + (A_2x \\
&\quad + B_2)(x^2 + 1).
\end{aligned}$$

比较等式两端 x 的同次幂的系数, 得

$$\begin{array}{l|l}
x^3 & A_1 + A_2 = 0, \\
x^2 & B_1 + B_2 = -5, \\
x^1 & 4A_1 + A_2 = 0, \\
x^0 & 4B_1 + B_2 = -4.
\end{array}$$

由此, $A_1 = 0, B_1 = \frac{1}{3}, A_2 = 0, B_2 = -\frac{16}{3}.$

于是,

$$\begin{aligned}
& \int \frac{x^4}{x^4 + 5x^2 + 4} dx \\
&= \int \left[1 + \frac{1}{3(x^2 + 1)} - \frac{16}{3(x^2 + 4)} \right] dx \\
&= x + \frac{1}{3} \operatorname{arctg} x - \frac{8}{3} \operatorname{arctg} \frac{x}{2} + C.
\end{aligned}$$

$$1871. \int \frac{x dx}{x^3 - 3x + 2}.$$

解 $\frac{x}{x^3 - 3x + 2} = \frac{x}{(x-1)^2(x+2)}$
 $= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$, 通分后应有
 $x \equiv A(x-1)(x+2) + B(x+2) + C(x-1)^2$.
 在这恒等式中,

$$\text{令 } x = 1, \text{ 得 } 1 = 3B, B = \frac{1}{3};$$

$$\text{令 } x = -2, \text{ 得 } -2 = 9C, C = -\frac{2}{9},$$

$$\text{比较 } x^2 \text{ 的系数, 得 } A + C = 0, \text{ 从而 } A = \frac{2}{9}.$$

于是,

$$\begin{aligned} \int \frac{x dx}{x^3 - 3x + 2} &= \int \left(\frac{2}{9(x-1)} \right. \\ &\quad \left. + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)} \right) dx \\ &= -\frac{1}{3(x-1)} + \frac{2}{9} \ln \left| \frac{x-1}{x+2} \right| + C. \end{aligned}$$

$$1872. \int \frac{x^2 + 1}{(x+1)^2(x-1)} dx.$$

解 设 $\frac{x^2 + 1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$, 通分后应有

$$\begin{aligned} x^2 + 1 &\equiv A(x+1)(x-1) + B(x-1) \\ &\quad + C(x+1)^2. \end{aligned}$$

在这恒等式中,

令 $x = -1$, 得 $2 = -2B, B = -1$;

令 $x = 1$, 得 $2 = 4C, C = \frac{1}{2}$;

比较 x^2 的系数, 得 $A + C = 1$, 从而 $A = \frac{1}{2}$.

于是,

$$\begin{aligned} & \int \frac{x^2 + 1}{(x+1)^2(x-1)} dx \\ &= \int \left(\frac{1}{2(x+1)} - \frac{1}{(x+1)^2} + \frac{1}{2(x+1)} \right) dx \\ &= \frac{1}{2} \ln |x^2 - 1| + \frac{1}{x+1} + C. \end{aligned}$$

1873. $\int \left(\frac{x}{x^2 - 3x + 2} \right)^2 dx.$

解 $\left(\frac{x}{x^2 - 3x + 2} \right)^2 = \frac{x^2}{(x-1)^2(x-2)^2}$
 $= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$, 通分后
应有

$$\begin{aligned} x^2 &\equiv A(x-1)(x-2)^2 + B(x-2)^2 \\ &\quad + C(x-2)(x-1)^2 + D(x-1)^2. \end{aligned}$$

在这恒等式中,

令 $x = 1$ 得 $B = 1$;

令 $x = 2$, 得 $D = 4$;

比较 x^3 及 x^2 的系数, 得

$$A + C = 0 \text{ 及 } -5A + B - 4C + D = 1;$$

由此, $A = 4, C = -4$.

于是,

$$\int \left(\frac{x}{x^2 - 3x + 2} \right)^2 dx$$

$$\begin{aligned}
&= \int \left(\frac{4}{x-1} + \frac{1}{(x-1)^2} - \frac{4}{x-2} + \frac{4}{(x-2)^2} \right) dx \\
&= 4\ln|x-1| - \frac{1}{x-1} - 4\ln|x-2| - \frac{4}{x-2} \\
&\quad + C \\
&= 4\ln \left| \frac{x-1}{x-2} \right| - \frac{5x-6}{x^2-3x+2} + C.
\end{aligned}$$

1874. $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}.$

解 设 $\frac{1}{(x+1)(x+2)^2(x+3)^3}$

$$\begin{aligned}
&= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{x+3} \\
&\quad + \frac{E}{(x+3)^2} + \frac{F}{(x+3)^3}, \text{通分后应有} \\
&1 \equiv A(x+2)^2(x+3)^3 \\
&\quad + B(x+1)(x+2)(x+3)^3 \\
&\quad + C(x+1)(x+3)^3 \\
&\quad + D(x+1)(x+2)^2(x+3)^2 \\
&\quad + E(x+1)(x+2)^2(x+3) \\
&\quad + F(x+1)(x+2)^2.
\end{aligned}$$

在这恒等式中,

令 $x = -1$, 得 $1 = 8A, A = \frac{1}{8};$

令 $x = -2$, 得 $1 = -C, C = -1;$

令 $x = -3$, 得 $1 = -2F, F = -\frac{1}{2}.$

比较 x^5, x^4 及 x^3 的系数, 得

$$\text{令 } x = 1, \text{得 } 1 = 8B, B = \frac{1}{8}.$$

$$\text{令 } x = -1, \text{得 } 1 = 4E, E = \frac{1}{4}.$$

$$\text{令 } x = 0, \text{得 } -A + B + C + D + E = 1;$$

$$\text{令 } x = 2, \text{得 } 27A + 27B + 9C + 3D + E = 1;$$

$$\text{令 } x = -2, \text{得 } 3A - B + 9C - 9D + 9E = 1;$$

$$\text{由此, } A = -\frac{3}{16}, C = \frac{3}{16}, D = \frac{1}{4}.$$

于是,

$$\begin{aligned} & \int \frac{dx}{x^5 + x^4 - 2x^3 - 2x^2 + x + 1} \\ &= \int \left[-\frac{3}{16(x-1)} + \frac{1}{8(x-1)^2} + \frac{3}{16(x+1)} \right. \\ & \quad \left. + \frac{1}{4(x+1)^2} + \frac{1}{4(x+1)^3} \right] dx \\ &= -\frac{3}{16} \ln|x-1| - \frac{1}{8(x-1)} + \frac{3}{16} \ln|x+1| \\ & \quad - \frac{1}{4(x+1)} - \frac{1}{8(x+1)} + C \\ &= \frac{3}{16} \ln \left| \frac{x+1}{x-1} \right| - \frac{3x^2 + 3x - 2}{8(x-1)(x+1)^2} + C. \end{aligned}$$

$$1876. \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx.$$

解 设 $\frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}$, 通分后

应有

$$\begin{aligned} x^2 + 5x + 4 &\equiv (Ax + B)(x^2 + 4) \\ &\quad + (Cx + D)(x^2 + 1). \end{aligned}$$

比较等式两端 x 的同次幂的系数, 得

$$\begin{array}{l|l} x^3 & A + C = 0, \\ x^2 & B + D = 1, \\ x^1 & 4A + C = 5, \\ x^0 & 4B + D = 4. \end{array}$$

由此, $A = \frac{5}{3}, B = 1, C = -\frac{5}{3}, D = 0$.

于是,

$$\begin{aligned} & \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx \\ &= \int \left(\frac{\frac{5}{3}x + 1}{x^2 + 1} + \frac{-\frac{5}{3}x}{x^2 + 4} \right) dx \\ &= \frac{5}{6} \ln \frac{x^2 + 1}{x^2 + 4} + \arctan x + C. \end{aligned}$$

本题如不直接用待定系数法将被积函数进行分解, 而使用其它技巧, 也可有更简单的方法. 事实上,

$$\begin{aligned} & \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx \\ &= \int \frac{x^2 + 4}{(x^2 + 1)(x^2 + 4)} dx + 5 \int \frac{x dx}{(x^2 + 4)(x^2 + 1)} \\ &= \int \frac{dx}{x^2 + 1} + \frac{5}{2} \int \frac{d(x^2)}{(x^2 + 4)(x^2 + 1)} \\ &= \arctan x + \frac{5}{6} \int \left(\frac{1}{x^2 + 1} - \frac{1}{x^2 + 4} \right) d(x^2) \\ &= \arctan x + \frac{5}{6} \ln \frac{x^2 + 1}{x^2 + 4} + C. \end{aligned}$$

1877. $\int \frac{dx}{(x+1)(x^2+1)}.$

解 设 $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$, 通分

后应有

$$1 \equiv A(x^2 + 1) + (Bx + C)(x + 1).$$

比较等式两端 x 的同次幂的系数, 得

$$\begin{array}{l|l} x^2 & A + B = 0, \\ x^1 & B + C = 0, \\ x^0 & A + C = 1. \end{array}$$

$$\text{由此, } A = \frac{1}{2}, B = -\frac{1}{2}, C = \frac{1}{2}.$$

于是,

$$\begin{aligned} & \int \frac{dx}{(x+1)(x^2+1)} \\ &= \int \left(\frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)} \right) dx \\ &= \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan x + C \\ &= \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} + \frac{1}{2} \arctan x + C. \end{aligned}$$

$$1878. \int \frac{dx}{(x^2 - 4x + 4)(x^2 - 4x + 5)}.$$

$$\begin{aligned} \text{解} \quad & \text{由于 } \frac{1}{(x^2 - 4x + 4)(x^2 - 4x + 5)} \\ &= \frac{(x^2 - 4x + 5) - (x^2 - 4x + 4)}{(x^2 - 4x + 4)(x^2 - 4x + 5)} \\ &= \frac{1}{(x-2)^2} - \frac{1}{x^2 - 4x + 5}, \end{aligned}$$

于是,

$$\begin{aligned} & \int \frac{dx}{(x^2 - 4x + 4)(x^2 - 4x + 5)} \\ &= \int \left(\frac{1}{(x-2)^2} - \frac{1}{x^2 - 4x + 5} \right) dx \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{x-2} - \int \frac{d(x-2)}{(x-2)^2 + 1} \\
&= -\frac{1}{x-2} - \arctan(x-2) + C.
\end{aligned}$$

本题若用待定系数法,较麻烦一些,也可获得同样的结果,此处从略.

1879. $\int \frac{x dx}{(x-1)^2(x^2+2x+2)}.$

解 设 $\frac{x}{(x-1)^2(x^2+2x+2)}$
 $= \frac{A}{x-1} + \frac{B}{(x-1)^2}$
 $+ \frac{Cx+D}{x^2+2x+2}$, 通分后应有

$$\begin{aligned}
x &\equiv A(x-1)(x^2+2x+2) + B(x^2+2x+2) \\
&\quad + (Cx+D)(x-1)^2.
\end{aligned}$$

比较等式两端 x 的同次幂系数,得

$$\begin{array}{l|l}
x^3 & A + C = 0, \\
x^2 & A + B - 2C + D = 0, \\
x^1 & 2B + C - 2D = 1, \\
x^0 & -2A + 2B + D = 0.
\end{array}$$

由此, $A = \frac{1}{25}, B = \frac{1}{5}, C = -\frac{1}{25}, D = -\frac{8}{25}$.

于是,

$$\begin{aligned}
&\int \frac{x dx}{(x-1)^2(x^2+2x+2)} \\
&= \int \left[\frac{1}{25(x-1)} + \frac{1}{5(x-1)^2} - \frac{x+8}{25(x^2+2x+2)} \right] dx \\
&= \frac{1}{25} \ln|x-1| - \frac{1}{5(x-1)} - \frac{1}{50} \int \frac{2x+2}{x^2+2x+2} dx
\end{aligned}$$

$$\begin{aligned}
& -\frac{7}{25} \int \frac{d(x+1)}{(x+1)^2+1} \\
& = \frac{1}{25} \ln|x-1| - \frac{1}{5(x-1)} - \frac{1}{50} \ln(x^2+2x+2) \\
& \quad - \frac{7}{25} \arctg(x+1) + C \\
& = \frac{1}{50} \ln \frac{(x-1)^2}{x^2+2x+2} - \frac{1}{5(x-1)} \\
& \quad - \frac{7}{25} \arctg(x+1) + C.
\end{aligned}$$

1880. $\int \frac{dx}{x(1+x)(1+x+x^2)}.$

解 设 $\frac{1}{x(1+x)(1+x+x^2)}$
 $= \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1},$

通分后应有

$$\begin{aligned}
1 &= A(x+1)(1+x+x^2) + Bx(1+x+x^2) \\
& \quad + x(x+1)(Cx+D).
\end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l}
x^3 & A+B+C=0, \\
x^2 & 2A+B+C+D=0, \\
x^1 & 2A+B+D=0, \\
x^0 & A=1.
\end{array}$$

由此, $A=1, B=-1, C=0, D=-1.$

于是,

$$\begin{aligned}
& \int \frac{dx}{x(1+x)(1+x+x^2)} \\
& = \int \left(\frac{1}{x} - \frac{1}{1+x} - \frac{1}{1+x+x^2} \right) dx
\end{aligned}$$

$$= \ln \left| \frac{x}{1+x} \right| - \frac{2}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C.$$

本题也可以不用待定系数法. 事实上,

$$\begin{aligned} & \frac{1}{x(1+x)(1+x+x^2)} \\ &= \frac{1}{(x+x^2)(1+x+x^2)} \\ &= \frac{1}{x+x^2} - \frac{1}{1+x+x^2} \\ &= \frac{1}{x} - \frac{1}{1+x} - \frac{1}{1+x+x^2}. \end{aligned}$$

1881. $\int \frac{dx}{x^3+1}.$

解 设 $\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$, 通分后应有

$$1 \equiv A(x^2-x+1) + (Bx+C)(x+1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^2 & A+B=0, \\ x^1 & -A+B+C=0, \\ x^0 & A+C=1. \end{array}$$

由此, $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{2}{3}$,

于是,

$$\begin{aligned} \int \frac{dx}{x^3+1} &= \int \left(\frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)} \right) dx \\ &= \frac{1}{3} \int \frac{dx}{x+1} - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx \\ &\quad + \frac{1}{2} \int \frac{d(x - \frac{1}{2})}{(x - \frac{1}{2})^2 + \frac{3}{4}} \end{aligned}$$

$$= \frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arc\,tg} \frac{2x-1}{\sqrt{3}} + C.$$

1882. $\int \frac{x dx}{x^3-1}.$

解 设 $\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$, 通分后应有

$$x \equiv A(x^2+x+1) + (Bx+C)(x-1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^2 & A+B=0, \\ x^1 & A-B+C=1, \\ x^0 & A-C=0 \end{array}$$

由此, $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{1}{3}.$

于是,

$$\begin{aligned} \int \frac{x}{x^3-1} dx &= \int \left[\frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)} \right] dx \\ &= \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx \\ &\quad + \frac{1}{2} \int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} + \frac{1}{\sqrt{3}} \operatorname{arc\,tg} \frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$

1883. $\int \frac{dx}{x^4-1}.$

解 $\int \frac{dx}{x^4-1} = \frac{1}{2} \int \left(\frac{1}{x^2-1} - \frac{1}{x^2+1} \right) dx$
 $= \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \operatorname{arc\,tg} x + C.$

本题若用待定系数法,则较麻烦,从略.

1884. $\int \frac{dx}{x^4 + 1}$

解 本题如用待定系数法来作,主要步骤如下:

$$\text{设 } \frac{1}{x^4 + 1} = \frac{Ax + B}{x^2 + x\sqrt{2} + 1} + \frac{Cx + D}{x^2 - x\sqrt{2} + 1},$$

$$\text{则经计算可求得 } A = \frac{\sqrt{2}}{4}, B = \frac{1}{2}, C = -\frac{\sqrt{2}}{4},$$

$$D = \frac{1}{2}. \text{ 于是,}$$

$$\begin{aligned} \int \frac{dx}{x^4 + 1} &= \int \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2 + x\sqrt{2} + 1} dx \\ &+ \int \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2 - x\sqrt{2} + 1} dx \\ &= \frac{\sqrt{2}}{4} \int \frac{(x + \frac{\sqrt{2}}{2}) dx}{(x + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \\ &+ \frac{1}{4} \int \frac{dx}{(x + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \\ &- \frac{\sqrt{2}}{4} \int \frac{(x - \frac{\sqrt{2}}{2}) dx}{(x - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \\ &+ \frac{1}{4} \int \frac{dx}{(x - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\sqrt{2}} [\ln(x^2 + x\sqrt{2} + 1) - \ln(x^2 - x\sqrt{2} + 1)] \\
&\quad + \frac{\sqrt{2}}{4} \left[\operatorname{arc\,tg} \left(\frac{2x + \sqrt{2}}{\sqrt{2}} \right) \right. \\
&\quad \left. + \operatorname{arc\,tg} \left(\frac{2x - \sqrt{2}}{\sqrt{2}} \right) \right] + C \\
&= \frac{1}{4\sqrt{2}} \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} \\
&\quad + \frac{\sqrt{2}}{4} \operatorname{arc\,tg} \left(\frac{x\sqrt{2}}{1 - x^2} \right) + C.
\end{aligned}$$

如应用下列解法,则更简单些.

$$\begin{aligned}
\int \frac{dx}{x^4 + 1} &= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 1} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1} dx \\
&= \frac{1}{2\sqrt{2}} \operatorname{arc\,tg} \left(\frac{x^2 - 1}{x\sqrt{2}} \right)^{*}) \\
&\quad - \frac{1}{4\sqrt{2}} \ln \frac{x^2 - x\sqrt{2} + 1^{**})}{x^2 + x\sqrt{2} + 1} + C_1,
\end{aligned}$$

注意到 $\operatorname{arc\,tg} \left(\frac{x^2 - 1}{x\sqrt{2}} \right) = \frac{\pi}{2} + \operatorname{arc\,tg} \left(\frac{x\sqrt{2}}{1 - x^2} \right)$, 最后
 即得

$$\begin{aligned}
\int \frac{dx}{x^4 + 1} &= \frac{1}{2\sqrt{2}} \operatorname{arc\,tg} \left(\frac{x\sqrt{2}}{1 - x^2} \right) \\
&\quad + \frac{1}{4\sqrt{2}} \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} + C.
\end{aligned}$$

*) 利用 1712 题的结果.

**) 利用 1713 题的结果.

1885. $\int \frac{dx}{x^4 + x^2 + 1}.$

$$\begin{aligned}
& + C_1 \\
& = \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + \frac{1}{2\sqrt{3}} \arctan \left(\frac{\sqrt{3}x}{1-x^2} \right) + C_1 \\
& = \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + \frac{1}{2\sqrt{3}} \arctan \left(\frac{x^2 - 1}{x\sqrt{3}} \right) + C.
\end{aligned}$$

如不用待定系数法解本题,则更简单些,解法与上题类似:

$$\begin{aligned}
& \int \frac{dx}{x^4 + x^2 + 1} \\
& = \frac{1}{2} \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx \\
& = \frac{1}{2} \int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \\
& = \frac{1}{2} \int \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + 3} - \frac{1}{2} \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 1} \\
& = \frac{1}{2\sqrt{3}} \arctan \left(\frac{x^2 - 1}{x\sqrt{3}} \right) + \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + C.
\end{aligned}$$

1886. $\int \frac{dx}{x^6 + 1}.$

解 本题如用待定系数法来作,运算较麻烦,经计算可得

$$\begin{aligned}
\frac{1}{x^6 + 1} & = \frac{1}{3(x^2 + 1)} + \frac{\frac{\sqrt{3}}{6}x + \frac{1}{3}}{x^2 + x\sqrt{3} + 1} \\
& \quad + \frac{-\frac{\sqrt{3}}{6}x + \frac{1}{3}}{x^2 - x\sqrt{3} + 1},
\end{aligned}$$

积分步骤与 1884 题与 1885 题完全类似,不再详解,其

结果为 $\frac{1}{2}\operatorname{arctg}x + \frac{1}{6}\operatorname{arctg}(x^3)$

$$+ \frac{1}{4\sqrt{3}}\ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + C.$$

本题如不用待定系数法来作,则更简单些.下面列举两种解法:

解法一:

$$\begin{aligned} \int \frac{dx}{x^6 + 1} &= \frac{1}{2} \int \frac{x^4 + 1}{x^6 + 1} dx - \frac{1}{2} \int \frac{x^4 - 1}{x^6 + 1} dx \\ &= \frac{1}{2} \int \frac{x^2 + (x^4 - x^2 + 1)}{x^6 + 1} dx \\ &\quad - \frac{1}{2} \int \frac{(x^2 - 1)(x^2 + 1)}{(x^2 + 1)(x^4 - x^2 + 1)} dx \\ &= \frac{1}{2} \int \frac{x^2}{x^6 + 1} dx + \frac{1}{2} \int \frac{dx}{1 + x^2} - \frac{1}{2} \int \frac{x^2 - 1}{x^4 - x^2 + 1} dx \\ &= \frac{1}{6} \int \frac{d(x^3)}{1 + (x^3)^2} + \frac{1}{2} \operatorname{arctg}x - \frac{1}{2} \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 3} \\ &= \frac{1}{6} \operatorname{arctg}(x^3) + \frac{1}{2} \operatorname{arctg}x + \frac{1}{4\sqrt{3}} \ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - \sqrt{3} + 1} \\ &\quad + C. \end{aligned}$$

解法二:

仿照 1881 题的分解法,可得

$$\frac{1}{x^6 + 1} = \frac{1}{3(x^2 + 1)} - \frac{x^2 - 2}{3(x^4 - x^2 + 1)}.$$

于是,

$$\int \frac{dx}{x^6 + 1} = \frac{1}{3} \int \frac{dx}{x^2 + 1} - \frac{1}{3} \int \frac{(x^2 - 2)dx}{x^4 - x^2 + 1}$$

$$\begin{aligned}
&= \frac{1}{3} \operatorname{arctg} x - \frac{1}{6} \int \frac{(x^2 + 1) + (x^2 - 1)}{x^4 - x^2 + 1} dx \\
&\quad + \frac{1}{3} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 - x^2 + 1} dx \\
&= \frac{1}{3} \operatorname{arctg} x + \frac{1}{6} \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx \\
&\quad - \frac{1}{2} \int \frac{x^2 - 1}{x^4 - x^2 + 1} dx \\
&= \frac{1}{3} \operatorname{arctg} x + \frac{1}{6} \int \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + 1} - \frac{1}{2} \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 3} \\
&= \frac{1}{3} \operatorname{arctg} x + \frac{1}{6} \operatorname{arctg} \left(\frac{x^2 - 1}{x} \right) \\
&\quad + \frac{1}{4\sqrt{3}} \ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + C.
\end{aligned}$$

两种答案形式不同,实质上是一致的.

1887. $\int \frac{dx}{(1+x)(1+x^2)(1+x^3)}.$

解 设 $\frac{1}{(1+x)(1+x^2)(1+x^3)} = \frac{A}{x+1}$
 $+ \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2-x+1},$

通分后应有

$$\begin{aligned}
1 &\equiv A(x+1)(x^2+1)(x^2-x+1) + B(x^2+1) \\
&\quad (x^2-x+1) + (Cx+D)(x+1)^2(x^2-x+1) \\
&\quad + (Ex+F)(x+1)^2(x^2+1).
\end{aligned}$$

比较等式两端 x 的同次幂系数,得

$$\begin{array}{l|l}
x^5 & A + C + E = 0, \\
x^4 & B + C + D + 2E + F = 0, \\
x^3 & A + D + 2E + 2F - B = 0, \\
x^2 & A + 2B + C + 2E + 2F = 0, \\
x^1 & -B + C + D + E + 2F = 0, \\
x^0 & A + B + D + F = 1.
\end{array}$$

由此, $A = \frac{1}{3}$, $B = \frac{1}{6}$, $C = 0$, $D = \frac{1}{2}$, $E = -\frac{1}{3}$, $F = 0$.

于是,

$$\begin{aligned}
& \int \frac{dx}{(1+x)(1+x^2)(1+x^3)} \\
&= \int \left[\frac{1}{3(x+1)} + \frac{1}{6(x+1)^2} \right. \\
&\quad \left. + \frac{1}{2(x^2+1)} - \frac{x}{3(x^2-x+1)} \right] dx \\
&= \frac{1}{3} \ln|1+x| - \frac{1}{6(x+1)} + \frac{1}{2} \operatorname{arctg} x \\
&\quad - \frac{1}{6} \int \frac{(2x-1)dx}{x^2-x+1} - \frac{1}{6} \int \frac{d\left(x - \frac{1}{2}\right)}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} - \frac{1}{6(x+1)} + \frac{1}{2} \operatorname{arctg} x \\
&\quad - \frac{1}{3\sqrt{3}} \operatorname{arctg} \left(\frac{2x-1}{\sqrt{3}} \right) + C.
\end{aligned}$$

1888. $\int \frac{dx}{x^5 - x^4 + x^3 - x^2 + x - 1}.$

解 设 $\frac{1}{x^5 - x^4 + x^3 - x^2 + x - 1}$

$$\begin{aligned}
&= \frac{1}{(x-1)(x^2-x+1)(x^2+x+1)} \\
&= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{Cx+D}{x^2-x+1},
\end{aligned}$$

通分后应有

$$\begin{aligned}
1 &\equiv A(x^2+x+1)(x^2-x+1) + (Bx+C)(x-1)(x^2-x+1) \\
&\quad + (Dx+E)(x-1)(x^2+x+1).
\end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l}
x^4 & A+B+D=0, \\
x^3 & -2B+C+E=0, \\
x^2 & A+2B-2C=0 \\
x^1 & -B+2C-D=0, \\
x^0 & A-C-E=1.
\end{array}$$

因此, $A = \frac{1}{3}, B = -\frac{1}{3}, C = -\frac{1}{6}, D = 0, E = -\frac{1}{2}$,
于是,

$$\begin{aligned}
&\int \frac{dx}{x^5-x^4+x^3-x^2+x-1} \\
&= \int \left[\frac{1}{3(x-1)} - \frac{2x+1}{6(x^2+x+1)} \right. \\
&\quad \left. - \frac{1}{2(x^2-x+1)} \right] dx \\
&= \frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}} \operatorname{arctg} \left(\frac{2x-1}{\sqrt{3}} \right) + C.
\end{aligned}$$

1889. $\int \frac{x^2 dx}{x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1}.$

解 设 $\frac{x^2}{x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{Cx + D}{x^2 + x + \frac{1}{2}}$, 通分后应有

$$x^2 \equiv (Ax + B)(x^2 + x + \frac{1}{2}) + (Cx + D)(x^2 + 2x + 2).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^3 & A + C = 0, \\ x^2 & A + B + 2C + D = 1, \\ x^1 & \frac{A}{2} + B + 2C + 2D = 0, \\ x^0 & \frac{B}{2} + 2D = 0. \end{array}$$

由此, $A = \frac{4}{5}, B = \frac{12}{5}, C = -\frac{4}{5}, D = -\frac{3}{5}$.

于是,

$$\begin{aligned} & \int \frac{x^2 dx}{x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1} \\ &= \int \left[\frac{4(x+3)}{5(x^2 + 2x + 2)} - \frac{4x+3}{5(x^2 + x + \frac{1}{2})} \right] dx \\ &= \frac{2}{5} \int \frac{(2x+2)dx}{x^2 + 2x + 2} + \frac{8}{5} \int \frac{d(x+1)}{(x+1)^2 + 1} \\ & \quad - \frac{2}{5} \int \frac{(2x+1)dx}{x^2 + x + \frac{1}{2}} - \frac{1}{5} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \frac{1}{4}} \end{aligned}$$

$$= \frac{2}{5} \ln \frac{x^2 + 2x + 2}{x^2 + x + \frac{1}{2}} + \frac{8}{5} \operatorname{arctg}(x + 1) \\ - \frac{2}{5} \operatorname{arctg}(2x + 1) + C.$$

1890. 在什么条件下, 积分

$$\int \frac{ax^2 + bx + c}{x^3(x-1)^2} dx$$

为有理函数?

解 设 $\frac{ax^2 + bx + c}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2},$

通分后应有

$$ax^2 + bx + c \equiv Ax^2(x-1)^2 + Bx(x-1)^2 \\ + C(x-1)^2 + Dx^3(x-1) + Ex^3.$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^4 & A + D = 0, \\ x^3 & -2A + B - D + E = 0, \\ x^2 & A - 2B + C = a \\ x^1 & B - 2C = b, \\ x^0 & C = c. \end{array}$$

由此, $A = a + 2b + 3c, B = b + 2c, C = c,$

$$D = -(a + 2b + 3c), E = a + b + c.$$

当 $A = D = 0$, 即 $a + 2b + 3c = 0$ 时, 积分

$$\int \frac{ax^2 + bx + c}{x^3(x-1)^2} dx$$

为有理函数.

利用奥斯特洛格拉得斯基方法*, 计算积分:

$$1891. \int \frac{x dx}{(x-1)^2(x+1)^3}.$$

$$\text{解 } Q = (x-1)^2(x+1)^3,$$

$$\begin{aligned} Q_1 &= (x-1)(x+1)^2 \\ &= x^3 + x^2 - x - 1, \end{aligned}$$

$$\begin{aligned} Q_2 &= (x-1)(x+1) \\ &= x^2 - 1. \end{aligned}$$

$$\text{设 } \frac{x}{(x-1)^2(x+1)^3}$$

$$= \left(\frac{Ax^2 + Bx + C}{x^3 + x^2 - x - 1} \right)' + \frac{Dx + E}{x^2 - 1}, \text{ 从而}$$

$$x \equiv (2Ax + B)(x-1)(x+1) - (3x-1)$$

$$+ (Ax^2 + Bx + C) + (Dx + E)(x-1)(x+1)^2.$$

比较等式两端 x 的同次幂系数, 得

* 所谓奥氏方法, 是指关于有理真分式 $\frac{P(x)}{Q(x)}$ 的积分, 可以借助代数方法来分离成一个真分式与另一个真分式积分的和, 使得在新的被积真分式函数中, 其分母次数达到最低状态, 也即在公式

$$\int \frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)} dx \quad (1)$$

中, 如果 $P(x), Q(x)$ 已知, 且设分母 $Q(x)$ 可以分解成一次与二次类型的实因式:

$$Q(x) = (x-a)^k \cdots (x^2 + px + q)^m \cdots,$$

其中 k, \cdots, m, \cdots 是自然数. 在公式(1)的右端分母已知, 形如:

$$Q_1(x) = (x-a)^{k-1} \cdots (x^2 + px + q)^{m-1} \cdots,$$

$$Q_2(x) = (x-a) \cdots (x^2 + px + q) \cdots,$$

且满足 $Q_1(x) \cdot Q_2(x) = Q(x)$. 而 $P_1(x)$ 和 $P_2(x)$ 为相应比 $Q_1(x)$ 和 $Q_2(x)$ 更低次的多项式, 一般可用待定系数法求得. 这种利用公式(1)来求积分的方法, 就是所谓的奥斯特洛格拉得斯基方法. 详细可以参见 Г. М. 菲赫金哥尔茨著(北大译), 微积分学教程, 第二卷一分册, 第 264 目. —《题解》编者注

$$\begin{array}{l|l} x^4 & D = 0, \\ x^3 & -A + D + E = 0, \\ x^2 & A - 2B - D + E = 0, \\ x^1 & -2A - 3C + B - D - E = 1, \\ x^0 & -B + C - E = 0. \end{array}$$

由此, $A = -\frac{1}{8}$, $B = -\frac{1}{8}$, $C = -\frac{1}{4}$, $D = 0$,

$$E = -\frac{1}{8}.$$

于是,

$$\begin{aligned} & \int \frac{x dx}{(x-1)^2(x+1)^3} \\ &= -\frac{x^2 + x + 2}{8(x-1)(x+1)^2} - \frac{1}{8} \int \frac{dx}{x^2 - 1} \\ &= -\frac{x^2 + x + 2}{8(x-1)(x+1)^2} + \frac{1}{16} \ln \left| \frac{x+1}{x-1} \right| + C. \end{aligned}$$

1892. $\int \frac{dx}{(x^3+1)^2}.$

解 $Q = (x+1)^2(x^2-x+1)^2,$

$$Q_1 = Q_2 = x^3 + 1.$$

设 $\frac{1}{(x^3+1)^2} = \left(\frac{Ax^2+Bx+C}{x^3+1} \right)' + \frac{Dx^2+Ex+F}{x^3+1},$ 从而

$$\begin{aligned} 1 &\equiv (2Ax+B)(x^3+1) - 3x^2(Ax^2+Bx+C) \\ &\quad + (Dx^2+Ex+F)(x^3+1). \end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^5 & E = 0, \\ x^4 & -A + F = 0, \\ x^3 & -2B + 2E = 0, \\ x^2 & 3A - 3C + 2F = 0, \\ x^1 & 2B - 4D + E = 0, \\ x^0 & C + F = 1. \end{array}$$

由此, $A = \frac{3}{8}, B = 0, C = \frac{5}{8}, D = 0, E = 0, F = \frac{3}{8}$.

$$\begin{aligned} \int \frac{dx}{(x^2 + 1)^3} &= \frac{x(3x^2 + 5)}{8(x^2 + 1)^3} + \frac{3}{8} \int \frac{dx}{x^2 + 1} \\ &= \frac{x(3x^2 + 5)}{8(x^2 + 1)^3} + \frac{3}{8} \operatorname{arctg} x + C. \end{aligned}$$

1894. $\int \frac{x^2 dx}{(x^2 + 2x + 2)^2}$

解 $Q = (x^2 + 2x + 2)^2, Q_1 = Q_2 = x^2 + 2x + 2$.

设 $\frac{x^2}{(x^2 + 2x + 2)^2} = \left(\frac{Ax + B}{x^2 + 2x + 2} \right)' + \frac{Cx + D}{x^2 + 2x + 2}$,

从而

$$\begin{aligned} x^2 &\equiv A(x^2 + 2x + 2) - 2(x + 1)(Ax + B) \\ &\quad + (Cx + D)(x^2 + 2x + 2). \end{aligned}$$

比较 等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^3 & C = 0, \\ x^2 & -A + 2C + D = 1, \\ x^1 & -2B + 2C + 2D = 0, \\ x^0 & 2A - 2B + 2D = 0. \end{array}$$

由此, $A = 0, B = 1, C = 0, D = 1$.

于是,

$$\begin{aligned}
& \int \frac{x^2 dx}{(x^2 + 2x + 2)^2} = \frac{1}{x^2 + 2x + 2} \\
& + \int \frac{dx}{x^2 + 2x + 2} \\
& = \frac{1}{x^2 + 2x + 2} + \int \frac{d(x+1)}{(x+1)^2 + 1} \\
& = \frac{1}{x^2 + 2x + 2} + \operatorname{arctg}(x+1) + C.
\end{aligned}$$

本题如不用奥斯特洛格拉得斯基方法,则更容易得出上述结果.事实上,

$$\begin{aligned}
& \int \frac{x^2 dx}{(x^2 + 2x + 2)^2} \\
& = \int \frac{(x^2 + 2x + 2) - (2x + 2)}{(x^2 + 2x + 2)^2} dx \\
& = \int \frac{dx}{x^2 + 2x + 2} - \int \frac{(2x + 2)dx}{(x^2 + 2x + 2)^2} \\
& = \int \frac{d(x+1)}{(x+1)^2 + 1} - \int \frac{d(x^2 + 2x + 2)}{(x^2 + 2x + 2)^2} \\
& = \operatorname{arctg}(x+1) + \frac{1}{x^2 + 2x + 2} + C.
\end{aligned}$$

1895. $\int \frac{dx}{(x^4 + 1)^2}.$

解 $Q = (x^4 + 1)^2, Q_1 = Q_2 = x^4 + 1,$

设 $\frac{1}{(x^4 + 1)^2} = \left(\frac{Ax^3 + Bx^2 + Cx + D}{x^4 + 1} \right) + \frac{Ex^3 + Fx^2 + Gx + H}{x^4 + 1},$ 从而

$$\begin{aligned}
1 & \equiv (3Ax^2 + 2Bx + C)(x^4 + 1) - 4x^3(Ax^3 + Bx^2 \\
& + Cx + D) + (Ex^3 + Fx^2 + Gx + H)(x^4 + 1).
\end{aligned}$$

比较等式两端 x 的同次幂系数,得

$$\begin{array}{l|l}
x^7 & E = 0, \\
x^6 & -A + F = 0, \\
x^5 & -2B + G = 0, \\
x^4 & -3C + H = 0, \\
x^3 & -4D + E = 0, \\
x^2 & 3A + F = 0, \\
x^1 & 2B + G = 0, \\
x^0 & C + H = 1.
\end{array}$$

由此, $A = 0, B = 0, C = \frac{1}{4}, D = 0, E = 0, F = 0,$

$G = 0, H = \frac{3}{4}.$

于是,

$$\begin{aligned}
\int \frac{dx}{(x^4 + 1)^2} &= \frac{x}{4(x^4 + 1)} + \frac{3}{4} \int \frac{dx}{x^4 + 1} \\
&= \frac{x}{4(x^4 + 1)} + \frac{3}{16\sqrt{2}} \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} \\
&\quad - \frac{3}{8\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{x^2 - 1} + C
\end{aligned}$$

*) 利用 1884 题的结果.

1896. $\int \frac{x^2 + 3x - 2}{(x - 1)(x^2 + x + 1)^2} dx.$

解 $Q = (x - 1)(x^2 + x + 1)^2, Q_1 = x^2 + x + 1,$
 $Q_2 = (x - 1)(x^2 + x + 1).$

设 $\frac{x^2 + 3x - 2}{(x - 1)(x^2 + x + 1)^2} = \left(\frac{Ax + B}{x^2 + x + 1} \right)$
 $+ \frac{Cx^2 + Dx + E}{(x - 1)(x^2 + x + 1)},$ 从而

$$x^2 + 3x - 2 \equiv A(x-1)(x^2+x+1) - (2x+1) \\ \cdot (Ax+B)(x-1) + (Cx^2+Dx+E)(x^2+x+1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^4 & C = 0, \\ x^3 & -A + C + D = 0, \\ x^2 & A - 2B + C + D + E = 1, \\ x^1 & A + B + D + E = 3, \\ x^0 & -A + B + E = -2. \end{array}$$

由此, $A = \frac{5}{3}, B = \frac{2}{3}, C = 0, D = \frac{5}{3}, E = -1.$

再将 $\frac{\frac{5}{3}x - 1}{(x-1)(x^2+x+1)}$ 分解, 可得

$$\frac{\frac{5}{3}x - 1}{(x-1)(x^2+x+1)} = \frac{2}{9(x-1)} \\ - \frac{2x-11}{9(x^2+x+1)}.$$

于是,

$$\begin{aligned} & \int \frac{x^2 + 3x - 2}{(x-1)(x^2+x+1)^2} dx \\ &= \frac{5x+2}{3(x^2+x+1)} + \frac{2}{9} \int \frac{dx}{x-1} \\ & \quad - \frac{1}{9} \int \frac{2x-11}{x^2+x+1} dx \\ &= \frac{5x+2}{3(x^2+x+1)} + \frac{2}{9} \ln|x-1| \\ & \quad - \frac{1}{9} \int \frac{2x+1}{x^2+x+1} dx + \frac{4}{3} \int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \end{aligned}$$

$$= \frac{5x+2}{3(x^2+x+1)} + \frac{1}{9} \ln \frac{(x-1)^2}{x^2+x+1} \\ + \frac{8}{3\sqrt{3}} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C.$$

1897. $\int \frac{dx}{(x^4-1)^3}.$

解 $Q = (x^4-1)^3, Q_1 = (x^4-1)^2, Q_2 = x^4-1.$

设 $\frac{1}{(x^4-1)^3} = \left[\frac{P(x)}{(x^4-1)^2} \right]' + \frac{P_1(x)}{x^4-1},$ 其中

$$P(x) = Ax^7 + Bx^6 + Cx^5 + Dx^4 + Ex^3 \\ + Fx^2 + Gx + H,$$

$$P_1(x) = A_1x^3 + B_1x^2 + C_1x + D_1,$$

从而利用待定系数法, 解出 $A=0, B=0, C=\frac{7}{32},$

$$D=0, E=0, F=0, G=-\frac{11}{32}, H=0, A_1=0,$$

$$B_1=0, C_1=0, D_1=\frac{21}{32}.$$

于是,

$$\int \frac{dx}{(x^4-1)^3} = \frac{7x^5-11x}{32(x^4-1)^2} + \frac{21}{32} \int \frac{dx}{x^4-1} \\ = \frac{7x^5-11x}{32(x^4-1)^2} + \frac{21}{128} \ln \left| \frac{x-1}{x+1} \right| - \frac{21}{64} \operatorname{arctg} x'' + C.$$

*) 利用 1883 题的结果.

分出下列积分的代数部分:

1898. $\int \frac{x^2+1}{(x^4+x^2+1)^2} dx.$

解 设 $\int \frac{x^2+1}{(x^4+x^2+1)^2} dx$

$$= \frac{Ax^3 + Bx^2 + Cx + D}{x^4 + x^2 + 1} \\ + \int \frac{A_1x^3 + B_1x^2 + C_1x + D_1}{x^4 + x^2 + 1} dx.$$

上述等式右端的积分为非代数部分, 因此, 只需要求出 A, B, C, D 就可以了. 等式两端求导并通分, 得

$$x^2 + 1 \equiv (3Ax^2 + 2Bx + C)(x^4 + x^2 + 1) \\ - (4x^3 + 2x)(Ax^3 + Bx^2 + Cx + D) \\ + (A_1x^3 + B_1x^2 + C_1x + D_1)(x^4 + x^2 + 1).$$

比较等式两端 x 的同次幂系数, 解出 $A = \frac{1}{6}, B = 0,$
 $C = \frac{1}{3}, D = 0, A_1 = 0, B_1 = \frac{1}{6}, C_1 = 0, D_1 = \frac{2}{3}$. 因此, 所求积分的代数部分为

$$\frac{x^3 + 2x}{6(x^4 + x^2 + 1)}.$$

1899⁺. $\int \frac{dx}{(x^3 + x + 1)^3},$

解 设 $\int \frac{dx}{(x^3 + x + 1)^3}$
 $= \frac{Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F}{(x^3 + x + 1)^2}$
 $+ \int \frac{Gx^2 + Hx + L}{x^3 + x + 1} dx.$

对上述等式两端求导再通分, 得

$$1 \equiv (5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E) \\ (x^3 + x + 1) - 2(3x^2 + 1)(Ax^5 + Bx^4 \\ + Cx^3 + Dx^2 + Ex + F) + (Gx^2 + Hx + L) \\ (x^3 + x + 1)^2.$$

比较等式两端 x 的同次幂系数, 解出 $A = -\frac{243}{961}, B = \frac{357}{1922}, C = -\frac{405}{961}, D = -\frac{315}{1922}, E = \frac{156}{961}, F = -\frac{224}{961}, G = 0, H = -\frac{243}{961}, L = \frac{357}{961}$. 因此, 所求积分的代数部分为

$$\frac{486x^5 - 357x^4 + 810x^3 + 315x^2 - 312x + 448}{1922(x^3 + x + 1)^2}$$

1900. $\int \frac{4x^5 - 1}{(x^5 + x + 1)^2} dx.$

解 设 $\int \frac{4x^5 - 1}{(x^5 + x + 1)^2} dx$
 $= \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{x^5 + x + 1}$
 $+ \int \frac{Fx^4 + Gx^3 + Hx^2 + Lx + M}{x^5 + x + 1} dx.$

对上述等式两端求导再通分, 得

$$4x^5 - 1 \equiv (4Ax^3 + 3Bx^2 + 2Cx + D)(x^5 + x + 1) - (5x^4 + 1)(Ax^4 + Bx^3 + Cx^2 + Dx + E) + (Fx^4 + Gx^3 + Hx^2 + Lx + M)(x^5 + x + 1).$$

比较等式两端 x 的同次幂系数, 解出 $A = 0, B = 0, C = 0, D = -1, E = 0, F = 0, G = 0, H = 0, L = 0, M = 0$. 因此, 所求积分的代数部分为

$$-\frac{x}{x^5 + x + 1} \text{ (全部积分).}$$

1901. 计算积分

$$\int \frac{dx}{x^4 + 2x^3 + 3x^2 + 2x + 1}.$$

解 $Q = x^4 + 2x^3 + 3x^2 + 2x + 1 = (x^2 + x + 1)^2,$

$$Q_1 = Q_2 = x^2 + x + 1.$$

设 $\frac{1}{x^4 + 2x^3 + 3x^2 + 2x + 1}$

$$= \left(\frac{Ax + B}{x^2 + x + 1} \right)' + \frac{Cx + D}{x^2 + x + 1}, \text{ 从而}$$

$$1 \equiv A(x^2 + x + 1) - (2x + 1)(Ax + B) + (Cx + D)(x^2 + x + 1).$$

比较等式两端 x 的同次幂系数, 解出 $A = \frac{2}{3}, B = \frac{1}{3},$
 $C = 0, D = \frac{2}{3}.$ 于是,

$$\begin{aligned} & \int \frac{dx}{x^4 + 2x^3 + 3x^2 + 2x + 1} \\ &= \frac{2x + 1}{3(x^2 + x + 1)} + \frac{2}{3} \int \frac{dx}{x^2 + x + 1} \\ &= \frac{2x + 1}{3(x^2 + x + 1)} + \frac{2}{3} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{2x + 1}{3(x^2 + x + 1)} + \frac{4}{3\sqrt{3}} \operatorname{arctg}\left(\frac{2x + 1}{\sqrt{3}}\right) + C. \end{aligned}$$

1902⁺. 在甚么条件下, 积分

$$\int \frac{ax^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2} dx$$

为有理函数?

解 (1) 当 $a \neq 0$ 且 $b^2 - ac = 0$ 时, $ax^2 + 2bx + c = a(x - x_0)^2$, 其中 x_0 为实数. 此时

$$\frac{ax^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2}$$

$$= \frac{\alpha(x-x_0)^2 + 2\alpha x_0(x-x_0) + \alpha x_0^2 + 2\beta(x-x_0) + 2\beta x_0 + \gamma}{a^2(x-x_0)^4}$$

$$= \frac{\alpha}{a^2(x-x_0)^2} + \frac{2\alpha x_0 + 2\beta}{a^2(x-x_0)^3} + \frac{\alpha x_0^2 + 2\beta x_0 + \gamma}{a^2(x-x_0)^4}$$

从而积分为有理函数.

(2) 当 $a \neq 0$ 且 $b^2 - ac \neq 0$ 时, 则设

$$\begin{aligned} & \frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2} \\ &= \left(\frac{Ax + B}{ax^2 + 2bx + c} \right)' + \frac{Cx + D}{ax^2 + 2bx + c}, \end{aligned}$$

从而

$$\alpha x^2 + 2\beta x + \gamma \equiv A(ax^2 + 2bx + c) - (2ax + 2b)$$

$$(Ax + B) + (Cx + D)(ax^2 + 2bx + c).$$

比较等式两端 x 的同次幂系数, 可解得 $C = 0$,

$$D = \frac{2b\beta - a\gamma - ca}{2(b^2 - ac)}. \text{ 从而当 } a\gamma + ca = 2b\beta \text{ 时 } D = 0,$$

此时积分为有理函数.

(3) 当 $a = 0, b \neq 0$ 时,

$$\begin{aligned} & \frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2} \\ &= \frac{\alpha \left(x + \frac{c}{2b} \right)^2 - \frac{\alpha c}{b} \left(x + \frac{c}{2b} \right) + \frac{\alpha c^2}{4b^2} + 2\beta \left(x + \frac{c}{2b} \right) - \frac{\beta c}{b} + \gamma}{4b^2 \left(x + \frac{c}{2b} \right)^2} \\ &= \frac{\alpha}{4b^2} + \frac{2\beta - \frac{\alpha c}{b}}{4b^2 \left(x + \frac{c}{2b} \right)} + \frac{\frac{\alpha c^2}{4b^2} - \frac{\beta c}{b} + \gamma}{4b^2 \left(x + \frac{c}{2b} \right)^2}. \end{aligned}$$

1905. $\int \frac{x^3 dx}{x^8 + 3}$

解 $\int \frac{x^3 dx}{x^8 + 3} = \frac{1}{4} \int \frac{d(x^4)}{(x^4)^2 + 3}$
 $= \frac{1}{4\sqrt{3}} \arctan \left(\frac{x^4}{\sqrt{3}} \right) + C.$

1906. $\int \frac{x^2 + x}{x^6 + 1} dx.$

解 $\int \frac{x^2 + x}{x^6 + 1} dx = \frac{1}{3} \int \frac{d(x^3)}{(x^3)^2 + 1} + \frac{1}{2} \int \frac{d(x^2)}{(x^2)^3 + 1}$
 $= \frac{1}{3} \arctan(x^3) + \frac{1}{2} \left[\frac{1}{6} \ln \frac{(x^2 + 1)^2}{x^4 - x^2 + 1} \right. \\ \left. + \frac{1}{\sqrt{3}} \arctan \left(\frac{2x^2 - 1}{\sqrt{3}} \right) \right]^{*}) + C$
 $= \frac{1}{3} \arctan(x^3) + \frac{1}{12} \ln \frac{(x^2 + 1)^2}{x^4 - x^2 + 1} \\ + \frac{1}{2\sqrt{3}} \arctan \left(\frac{2x^2 - 1}{\sqrt{3}} \right) + C.$

*) 利用 1881 题的结果.

1907. $\int \frac{x^4 - 3}{x(x^8 + 3x^4 + 2)} dx.$

解 $\int \frac{x^4 - 3}{x(x^8 + 3x^4 + 2)} dx = \int \frac{\left(1 - \frac{3}{x^4}\right) dx}{x^5 \left(1 + \frac{3}{x^4} + \frac{2}{x^8}\right)}$
 $= \int \frac{-\frac{1}{4} \left(1 - \frac{3}{x^4}\right) d\left(\frac{1}{x^4}\right)}{\frac{2}{x^8} + \frac{3}{x^4} + 1}$
 $= -\frac{1}{4} \int \left[\frac{5}{\frac{2}{x^4} + 1} - \frac{4}{\frac{1}{x^4} + 1} \right] d\left(\frac{1}{x^4}\right)$

$$= -\frac{5}{8} \ln\left(\frac{2}{x^4} + 1\right) + \ln\left(\frac{1}{x^4} + 1\right) + C$$

$$= \frac{5}{8} \ln \frac{x^4}{x^4 + 2} - \ln \frac{x^4}{x^4 + 1} + C.$$

1908. $\int \frac{x^4 dx}{(x^{10} - 10)^2}.$

解 $\int \frac{x^4 dx}{(x^{10} - 10)^2}$

$$= \frac{1}{5} \int \frac{d(x^5)}{[(x^5 - \sqrt{10})(x^5 + \sqrt{10})]^2}$$

$$= \frac{1}{200} \int \frac{[(x^5 + \sqrt{10}) - (x^5 - \sqrt{10})]^2}{[(x^5 - \sqrt{10})(x^5 + \sqrt{10})]^2} d(x^5)$$

$$= \frac{1}{200} \int \left(\frac{1}{x^5 - \sqrt{10}} - \frac{1}{x^5 + \sqrt{10}} \right)^2 d(x^5)$$

$$= \frac{1}{200} \int \frac{d(x^5 - \sqrt{10})}{(x^5 - \sqrt{10})^2} - \frac{1}{100} \int \frac{d(x^5)}{(x^5)^2 - 10}$$

$$+ \frac{1}{200} \int \frac{d(x^5 + \sqrt{10})}{(x^5 + \sqrt{10})^2}$$

$$= -\frac{1}{200(x^5 - \sqrt{10})} - \frac{1}{200\sqrt{10}} \ln \left| \frac{x^5 - \sqrt{10}}{x^5 + \sqrt{10}} \right|$$

$$- \frac{1}{200(x^5 + \sqrt{10})} + C$$

$$= -\frac{1}{100} \left[\frac{x^5}{x^{10} - 10} + \frac{1}{2\sqrt{10}} \ln \left| \frac{x^5 - \sqrt{10}}{x^5 + \sqrt{10}} \right| \right] + C.$$

1909. $\int \frac{x^{11} dx}{x^8 + 3x^4 + 2}.$

解 $\int \frac{x^{11}}{x^8 + 3x^4 + 2} dx = \frac{1}{4} \int \frac{x^8 d(x^4)}{(x^4 + 1)(x^4 + 2)}$

$$= \frac{1}{4} \int \left(1 - \frac{3x^4 + 2}{(x^4 + 1)(x^4 + 2)} \right) d(x^4)$$

$$\begin{aligned}
&= \frac{1}{4} \int \left(1 + \frac{1}{x^4 + 1} - \frac{4}{x^4 + 2} \right) d(x^4) \\
&= \frac{x^4}{4} + \frac{1}{4} \ln \frac{x^4 + 1}{(x^4 + 2)^4} + C.
\end{aligned}$$

1910. $\int \frac{x^9 dx}{(x^{10} + 2x^5 + 2)^2}.$

解
$$\begin{aligned}
\int \frac{x^9 dx}{(x^{10} + 2x^5 + 2)^2} &= \frac{1}{5} \int \frac{x^5 d(x^5)}{[(x^5 + 1)^2 + 1]^2} \\
&= \frac{1}{5} \int \frac{(x^5 + 1) d(x^5 + 1)}{[(x^5 + 1)^2 + 1]^2} - \frac{1}{5} \int \frac{d(x^5 + 1)}{[(x^5 + 1)^2 + 1]^2} \\
&= \frac{1}{10} \int \frac{d[(x^5 + 1)^2 + 1]}{[(x^5 + 1)^2 + 1]^2} - \frac{1}{5} \int \frac{d(x^5 + 1)}{[(x^5 + 1)^2 + 1]^2} \\
&= -\frac{1}{10[(x^5 + 1)^2 + 1]} - \frac{1}{5} \left\{ \frac{x^5 + 1}{2[(x^5 + 1)^2 + 1]} \right. \\
&\quad \left. + \frac{1}{2} \operatorname{arc} \operatorname{tg}(x^5 + 1) \right\}'' + C \\
&= \frac{x^5 + 2}{10(x^{10} + 2x^5 + 2)} - \frac{1}{10} \operatorname{arc} \operatorname{tg}(x^5 + 1) + C.
\end{aligned}$$

*) 利用 1817 题的结果.

1911. $\int \frac{x^{2n-1}}{x^n + 1} dx.$

解 当 $n \neq 0$ 时,

$$\begin{aligned}
\int \frac{x^{2n-1}}{x^n + 1} dx &= \int \frac{x^n \cdot x^{n-1} dx}{x^n + 1} \\
&= \frac{1}{n} \int \frac{x^n d(x^n)}{x^n + 1} \\
&= \frac{1}{n} \int \left(1 - \frac{1}{x^n + 1} \right) d(x^n) \\
&= \frac{1}{n} (x^n - \ln|x^n + 1|) + C;
\end{aligned}$$

当 $n = 0$ 时,

$$\int \frac{x^{2n-1}}{x^n + 1} dx = \int \frac{dx}{2x} = \frac{1}{2} \ln |x| + C.$$

1912. $\int \frac{x^{3n-1}}{(x^{2n} + 1)^2} dx.$

解 当 $n \neq 0$ 时,

$$\begin{aligned} \int \frac{x^{3n-1}}{(x^{2n} + 1)^2} dx &= \int \frac{x^{2n} \cdot x^{n-1} dx}{(x^{2n} + 1)^2} \\ &= \frac{1}{n} \int \frac{x^{2n} d(x^n)}{(x^{2n} + 1)^2} \\ &= \frac{1}{n} \int \frac{(x^{2n} + 1) - 1}{(x^{2n} + 1)^2} d(x^n) \\ &= \frac{1}{n} \int \frac{d(x^n)}{x^{2n} + 1} - \frac{1}{n} \int \frac{d(x^n)}{(x^{2n} + 1)^2} \\ &= \frac{1}{n} \arctg(x^n) - \frac{1}{n} \left[\frac{x^n}{2(x^{2n} + 1)} \right. \\ &\quad \left. + \frac{1}{2} \arctg(x^n) \right]^{**} + C \\ &= \frac{1}{2n} \left[\arctg(x^n) - \frac{x^n}{x^{2n} + 1} \right] + C. \end{aligned}$$

当 $n = 0$ 时,

$$\int \frac{x^{3n-1}}{(x^{2n} + 1)^2} dx = \frac{1}{4} \int \frac{dx}{x} = \frac{1}{4} \ln |x| + C.$$

*) 利用 1817 题的结果.

1913. $\int \frac{dx}{x(x^{10} + 2)}.$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x(x^{10} + 2)} &= \frac{1}{2} \int \left(\frac{1}{x} - \frac{x^9}{x^{10} + 2} \right) dx \\ &= \frac{1}{2} \ln |x| - \frac{1}{20} \int \frac{d(x^{10} + 2)}{x^{10} + 2} \\ &= \frac{1}{2} \ln |x| - \frac{1}{20} \ln(x^{10} + 2) + C \end{aligned}$$

$$= \frac{1}{20} \ln \frac{x^{10}}{x^{10} + 2} + C.$$

1914. $\int \frac{dx}{x(x^{10} + 1)^2}.$

解 由于

$$\begin{aligned} \frac{1}{x(x^{10} + 1)^2} &= \frac{x^{10} + 1 - x^{10}}{x(x^{10} + 1)^2} \\ &= \frac{1}{x(x^{10} + 1)} - \frac{x^9}{(x^{10} + 1)^2} \\ &= \frac{1}{x} - \frac{x^9}{x^{10} + 1} - \frac{x^9}{(x^{10} + 1)^2}, \end{aligned}$$

所以

$$\begin{aligned} \int \frac{dx}{x(x^{10} + 1)^2} &= \int \left(\frac{1}{x} - \frac{x^9}{x^{10} + 1} - \frac{x^9}{(x^{10} + 1)^2} \right) dx \\ &= \ln|x| - \frac{1}{10} \int \frac{d(x^{10} + 1)}{x^{10} + 1} - \frac{1}{10} \int \frac{d(x^{10} + 1)}{(x^{10} + 1)^2} \\ &= \ln|x| - \frac{1}{10} \ln(x^{10} + 1) + \frac{1}{10(x^{10} + 1)} + C \\ &= \frac{1}{10} \ln \frac{x^{10}}{x^{10} + 1} + \frac{1}{10(x^{10} + 1)} + C. \end{aligned}$$

1915. $\int \frac{1 - x^7}{x(1 + x^7)} dx.$

解
$$\begin{aligned} \int \frac{1 - x^7}{x(1 + x^7)} dx &= \int \left(\frac{1}{x} - \frac{2x^6}{1 + x^7} \right) dx \\ &= \ln|x| - \frac{2}{7} \int \frac{d(1 + x^7)}{1 + x^7} \\ &= \ln|x| - \frac{2}{7} \ln|1 + x^7| + C \\ &= \frac{1}{7} \ln \frac{|x|^7}{(1 + x^7)^2} + C. \end{aligned}$$

1916. $\int \frac{x^4 - 1}{x(x^4 - 5)(x^5 - 5x + 1)} dx.$

$$\begin{aligned}
& \text{解} \quad \int \frac{x^4 - 1}{x(x^4 - 5)(x^5 - 5x + 1)} dx \\
&= \frac{1}{5} \int \frac{d(x^5 - 5x)}{(x^5 - 5x)(x^5 - 5x + 1)} \\
&= \frac{1}{5} \int \left(\frac{1}{x^5 - 5x} - \frac{1}{x^5 - 5x + 1} \right) d(x^5 - 5x) \\
&= \frac{1}{5} \int \frac{d(x^5 - 5x)}{x^5 - 5x} - \frac{1}{5} \int \frac{d(x^5 - 5x + 1)}{x^5 - 5x + 1} \\
&= \frac{1}{5} \ln \left| \frac{x(x^4 - 5)}{x^5 - 5x + 1} \right| + C.
\end{aligned}$$

$$1917. \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx.$$

解 由于

$$\begin{aligned}
\frac{x^2 + 1}{x^4 + x^2 + 1} &= \frac{x^2 + 1}{(x^2 + 1)^2 - x^2} \\
&= \frac{x^2 + 1}{(x^2 - x + 1)(x^2 + x + 1)} \\
&= \frac{1}{2} \left(\frac{1}{x^2 - x + 1} + \frac{1}{x^2 + x + 1} \right),
\end{aligned}$$

所以

$$\begin{aligned}
& \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx \\
&= \frac{1}{2} \int \frac{dx}{x^2 - x + 1} + \frac{1}{2} \int \frac{dx}{x^2 + x + 1} \\
&= \frac{1}{2} \int \frac{d\left(x - \frac{1}{2}\right)}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x - 1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x + 1}{\sqrt{3}} + C_1 \\
&= \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x^2 - 1}{x \sqrt{3}} + C.
\end{aligned}$$

$$1918. \int \frac{x^2 - 1}{x^4 + x^3 + x^2 + x + 1} dx$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{x^2 - 1}{x^4 + x^3 + x^2 + x + 1} dx \\
 &= \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1} \\
 &= \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1} \\
 &= \int \frac{d\left(x + \frac{1}{x} + \frac{1}{2}\right)}{\left[\left(x + \frac{1}{x}\right) + \frac{1}{2}\right]^2 - \frac{5}{4}} \\
 &= \frac{1}{\sqrt{5}} \ln \frac{x + \frac{1}{x} + \frac{1}{2} - \frac{\sqrt{5}}{2}}{x + \frac{1}{x} + \frac{1}{2} + \frac{\sqrt{5}}{2}} + C \\
 &= \frac{1}{\sqrt{5}} \ln \frac{2x^2 + (1 - \sqrt{5})x + 2}{2x^2 + (1 + \sqrt{5})x + 2} + C.
 \end{aligned}$$

$$1919. \int \frac{x^5 - x}{x^8 + 1} dx.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{x^5 - x}{x^8 + 1} dx = \frac{1}{2} \int \frac{(x^2)^2 - 1}{(x^2)^4 + 1} d(x^2) \\
 &= \frac{1}{4\sqrt{2}} \ln \frac{x^4 - x^2\sqrt{2} + 1^{*})}{x^4 + x^2\sqrt{2} + 1} + C.
 \end{aligned}$$

*) 利用 1713 题的结果.

$$1920. \int \frac{x^4 + 1}{x^6 + 1} dx$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x^4 + 1}{x^6 + 1} dx &= \int \frac{(x^4 - x^2 + 1) + x^2}{x^6 + 1} dx \\
 &= \int \frac{x^4 - x^2 + 1}{(x^2 + 1)(x^4 - x^2 + 1)} dx + \int \frac{x^2 dx}{x^6 + 1} \\
 &= \int \frac{1}{x^2 + 1} dx + \frac{1}{3} \int \frac{d(x^3)}{(x^3)^2 + 1} \\
 &= \operatorname{arctg} x + \frac{1}{3} \operatorname{arctg}(x^3) + C.
 \end{aligned}$$

1921. 试导出计算积分

$$I_n = \int \frac{dx}{(ax^2 + bx + c)^n} \quad (a \neq 0)$$

的递推公式.

利用这个公式计算

$$I_3 = \int \frac{dx}{(x^2 + x + 1)^3}.$$

解 由于

$$\begin{aligned}
 4a(ax^2 + bx + c) &= (2ax + b)^2 + (4ac - b^2) \\
 &= t^2 + \Delta, \text{ 其中 } t = 2ax + b, \Delta = 4ac - b^2. \text{ 于是}
 \end{aligned}$$

$$\begin{aligned}
 I_n &= \int \frac{dx}{(ax^2 + bx + c)^n} = \int \frac{(4a)^n dx}{[(2ax + b)^2 + \Delta]^n} \\
 &= 2^{2n-1} a^{n-1} \int \frac{dt}{(t^2 + \Delta)^n}.
 \end{aligned}$$

当 $\Delta \neq 0$ 时, 对于积分 $\int \frac{dt}{(t^2 + \Delta)^n}$ 施用分部积分法, 即有

$$\begin{aligned}
 \int \frac{dt}{(t^2 + \Delta)^n} &= \frac{t}{(t^2 + \Delta)^n} + 2n \int \frac{t^2 dt}{(t^2 + \Delta)^{n+1}} \\
 &= \frac{t}{(t^2 + \Delta)^n} + 2n \int \frac{(t^2 + \Delta) - \Delta}{(t^2 + \Delta)^{n+1}} dt \\
 &= \frac{t}{(t^2 + \Delta)^n} + 2n \int \frac{dt}{(t^2 + \Delta)^n} - 2n\Delta \int \frac{dt}{(t^2 + \Delta)^{n+1}}.
 \end{aligned}$$

若令 $\bar{I}_n = \int \frac{dt}{(t^2 + \Delta)^n}$, 则得

$$\bar{I}_n = \frac{t}{(t^2 + \Delta)^n} + 2n\bar{I}_n - 2n\Delta\bar{I}_{n+1},$$

$$\text{或 } \bar{I}_{n+1} = \frac{1}{2n\Delta} \cdot \frac{t}{(t^2 + \Delta)^n} + \frac{2n-1}{2n} \cdot \frac{1}{\Delta} \cdot \bar{I}_n,$$

$$\begin{aligned} \text{从而 } \bar{I}_n &= \frac{1}{2(n-1)\Delta} \cdot \frac{t}{(t^2 + \Delta)^{n-1}} \\ &\quad + \frac{2n-3}{2n-2} \cdot \frac{1}{\Delta} \bar{I}_{n-1}. \end{aligned}$$

代入 I_n , 即得

$$\begin{aligned} I_n &= 2^{2n-1} \cdot a^{n-1} \cdot \left\{ \frac{1}{2(n-1)\Delta} \cdot \frac{t}{(t^2 + \Delta)^{n-1}} \right. \\ &\quad \left. + \frac{2n-3}{2n-2} \cdot \frac{1}{\Delta} \cdot \bar{I}_{n-1} \right\} \\ &= 2^{2n-1} \cdot a^{n-1} \left\{ \frac{1}{2(n-1)\Delta} \cdot \right. \\ &\quad \frac{2ax+b}{(4a)^{n-1}(ax^2+bx+c)^{n-1}} + \frac{2n-3}{2n-2} \\ &\quad \left. \cdot \frac{1}{\Delta} \cdot \frac{2a}{(4a)^{n-1}} \int \frac{dx}{(ax^2+bx+c)^{n-1}} \right\} \\ &= \frac{1}{(n-1)\Delta} \cdot \frac{2ax+b}{(ax^2+bx+c)^{n-1}} + \frac{2n-3}{n-1} \cdot \\ &\quad \frac{2a}{\Delta} I_{n-1}, \end{aligned}$$

最后得递推公式

$$\begin{aligned} I_n &= \frac{1}{(n-1)\Delta} \cdot \frac{2ax+b}{(ax^2+bx+c)^{n-1}} + \frac{2n-3}{n-1} \cdot \\ &\quad \frac{2a}{\Delta} I_{n-1}. \end{aligned}$$

当 $\Delta = 0$ 时, 则有

代入 I , 即得

$$I = \frac{1}{(b-a)^{m+n-1}} \int \frac{(1-t)^{m+n-2}}{t^m} dt \quad (a \neq b).$$

将 $(1-t)^{m+n-2}$ 展开, 即可分项积分求得 I .

如果 $b = a$, 则

$$I = \int \frac{dx}{(x+a)^{m+n}} = \frac{1}{1-m-n} (x+a)^{1-m-n} + C.$$

令 $a = -2, b = 3, m = 2$ 及 $n = 3$, 并设 $t = \frac{x-2}{x+3}$, 即得

$$\begin{aligned} & \int \frac{dx}{(x-2)^2(x+3)^3} \\ &= \frac{1}{5^4} \int \frac{(1-t)^3}{t^2} dt \\ &= \frac{1}{5^4} \int \left(\frac{1}{t^2} - \frac{3}{t} + 3 - t \right) dt \\ &= \frac{1}{625} \left(-\frac{1}{t} - 3 \ln |t| + 3t - \frac{t^2}{2} \right) + C \\ &= \frac{1}{625} \left[-\frac{x+3}{x-2} - 3 \ln \left| \frac{x-2}{x+3} \right| + \frac{3(x-2)}{x+3} \right. \\ & \quad \left. - \frac{(x-2)^2}{2(x+3)^2} \right] + C. \end{aligned}$$

1923. 若 $P_n(x)$ 为 x 的 n 次多项式, 计算

$$\int \frac{P_n(x)}{(x-a)^{n+1}} dx.$$

解 由于 $P_n(x)$ 为 x 的 n 次多项式, 故得

$$P_n(x) = \sum_{k=0}^n \frac{P_n^{(k)}(a)}{k!} (x-a)^k,$$

其中 $P_n^{(0)}(a) = P_n(a), 0! = 1$.

于是,

$$\begin{aligned}
& \int \frac{P_n(x)}{(x-a)^{n+1}} dx \\
&= \sum_{k=0}^{n-1} \frac{1}{k!} P_n^{(k)}(a) \int \frac{dx}{(x-a)^{n-k+1}} \\
&\quad + \frac{1}{n!} P_n^{(n)}(a) \int \frac{dx}{x-a} \\
&= - \sum_{k=0}^{n-1} \frac{P_n^{(k)}(a)}{k!(n-k)(x-a)^{n-k}} \\
&\quad + \frac{1}{n!} P_n^{(n)}(a) \ln|x-a| + C,
\end{aligned}$$

其中 $\frac{P_n^{(n)}(a)}{n!} = a_0$ 为 $P_n(x)$ 的首项系数, 即

$$\begin{aligned}
P_n(x) &= a_0(x-a)^n + a_1(x-a)^{n-1} + \cdots \\
&\quad + a_{n-1}(x-a) + a_n,
\end{aligned}$$

1924⁺. 设 $R(x) = R^*(x^2)$, 其中 R^* 为有理函数, 则函数 $R(x)$ 分解为有理分式时有甚么特性?

解 设 $R^*(x) = P(x) + H(x)$,

其中 $P(x)$ 是多项式; 若 $R^*(x)$ 本身也为多项式时, 则 $H(x) \equiv 0$; 否则 $H(x) = \frac{P_1(x)}{Q_1(x)}$ 是真分式, 而 $P_1(x), Q_1(x)$ 也均为多项式.

设 $Q_1(x)$ 有非负实根为 a_i^2 , 其重数为 $\alpha_i (i = 1, 2, \cdots, m)$; 负根为 $-b_k^2$, 其重数为 $\beta_k (k = 1, 2, \cdots, t)$; 二次因式为 $x^2 + C_p x + D_p$, 其重数为 $\gamma_p (p = 1, 2, \cdots, s)$. 其中 $C_p^2 - 4D_p < 0$, 于是,

$$Q_1(x) = \begin{cases} a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i} \cdot \prod_{k=1}^t (x + b_k^2)^{\beta_k} \cdot \prod_{p=1}^s (x^2 + C_p x + D_p)^{\gamma_p}, & \text{当 } m \neq 0, t \neq 0, s \neq 0 \text{ 时;} \\ a_0 \prod_{k=1}^t (x + b_k^2)^{\beta_k} \cdot \prod_{p=1}^s (x^2 + C_p x + D_p)^{\gamma_p}, & \text{当 } m = 0, t \neq 0, s \neq 0 \text{ 时;} \\ a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i} \cdot \prod_{p=1}^s (x^2 + C_p x + D_p)^{\gamma_p}, & \text{当 } m \neq 0, t = 0, s \neq 0 \text{ 时;} \\ a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i} \cdot \prod_{k=1}^t (x + b_k^2)^{\beta_k}, & \text{当 } m \neq 0, t \neq 0, s = 0 \text{ 时;} \\ a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i}, & \text{当 } m \neq 0, t = 0, s = 0 \text{ 时;} \\ a_0 \prod_{k=1}^t (x + b_k^2)^{\beta_k}, & \text{当 } m = 0, t \neq 0, s = 0 \text{ 时;} \\ a_0 \prod_{p=1}^s (x^2 + C_p x + D_p)^{\gamma_p}, & \text{当 } m = 0, t = 0, s \neq 0 \text{ 时;} \end{cases}$$

以下就 $Q_1(x)$ 表达式中的第一种情形予以论证.

由 $C_p^2 - 4D_p < 0$, 必有

$$x^2 + C_p x + D_p = (x^2 + E_p x + F_p) \cdot (x^2 - E_p x + F_p) \\ (p = 1, 2, \dots, s), \text{ 则此时有}$$

$$Q_1(x^2) = a_0 \prod_{i=1}^m (x - a_i)^{\alpha_i} (x + a_i)^{\alpha_i} \cdot \prod_{k=1}^t (x^2 + b_k^2)^{\beta_k} \\ \cdot \prod_{p=1}^s (x^2 + E_p x + F_p)^{\gamma_p} (x^2 - E_p x + F_p)^{\gamma_p}, \text{ 以及}$$

$$H(x^2) = \frac{P_1(x^2)}{Q_1(x^2)} = \sum_{i=1}^m \sum_{\iota=1}^{u_i} \left(\frac{A_{i\iota}}{(a_i - x)^\iota} + \frac{A'_{i\iota}}{(a_i + x)^\iota} \right) \\ + \sum_{k=1}^t \sum_{\iota=1}^{\beta_k} \frac{B_{k\iota}x + C_{k\iota}}{(x^2 + b_k^2)^\iota} + \sum_{p=1}^s \sum_{\iota=1}^{\gamma_p} \left(\frac{M_{p\iota}x + N_{p\iota}}{(x^2 + E_px + F_p)^\iota} \right. \\ \left. + \frac{M'_{p\iota}x + N'_{p\iota}}{(x^2 - E_px + F_p)^\iota} \right).$$

显然有 $H(x^2) = H((-x)^2)$, 由 $H(x^2)$ 的分解式的唯一性, 比较系数, 即得常数关系为:

$$A'_{i\iota_1} = A_{i\iota_1}, M_{p\iota_2} = -M'_{p\iota_2}, N_{p\iota_2} = N'_{p\iota_2}, B_{k\iota_3} = 0, \\ (\iota_1 = 1, 2, \dots, u_i, i = 1, 2, \dots, m; \iota_2 = 1, 2, \dots, \gamma_p, p = 1, \\ 2, \dots, s; \iota_3 = 1, 2, \dots, \beta_k, k = 1, 2, \dots, t). \text{ 最后得} \\ R(x) = P(x^2) + H(x^2) =$$

$$= P(x^2) + \sum_{i=1}^m \sum_{\iota=1}^{u_i} A_{i\iota} \left(\frac{1}{(a_i - x)^\iota} + \frac{1}{(a_i + x)^\iota} \right) \\ + \sum_{k=1}^t \sum_{\iota=1}^{\beta_k} \frac{C_{k\iota}}{(x^2 + b_k^2)^\iota} + \sum_{p=1}^s \sum_{\iota=1}^{\gamma_p} \left(\frac{M_{p\iota}x + N_{p\iota}}{(x^2 + E_px + F_p)^\iota} \right. \\ \left. - \frac{M_{p\iota}x - N_{p\iota}}{(x^2 - E_px + F_p)^\iota} \right).$$

如若 $H(x) \neq 0$, 而 $m = 0$, 但 $t \neq 0, s \neq 0$ 时, 则在上述表达式中就应该缺乏第二项的和式, 形如

$$R(x) = P(x^2) + \sum_{k=1}^t \sum_{\iota=1}^{\beta_k} + \sum_{p=1}^s \sum_{\iota=1}^{\gamma_p},$$

其它情形可以类似推演, 此处不再一一细叙。至于当 $H(x) \equiv 0$ 时, 当然有 $R(x) = P(x^2)$ 。

另外, 本题也可在复数域上作分解考虑。

仍记 $R^*(x) = P(x) + H(x)$, 其中 $P(x)$ 为多项

式, 而 $H(x)$ 要么是零(当 $R^*(x)$ 为多项式时), 要么是一个真分式, 例如 $H(x) \neq 0$ 时, 记 $H(x) = \frac{P_1(x)}{Q_1(x)}$ 是其真分式. $P_1(x), Q_1(x)$ 为多项式. 若记 $Q_1(x)$ 在复数域中的根为 a_i , 其相应重数记为 $n_i (i = 1, 2, \dots, m; \text{显然 } m \geq 1)$. 即

$$Q_1(x) = a_0 \prod_{i=1}^m (x - a_i)^{n_i},$$

那么 $Q_1(x^2)$ 中的每一项 $x^2 - a_i$ 可分解为一次式乘积

$$x^2 - a_i = (x - b_i)(x + b_i),$$

于是

$$Q_1(x^2) = a_0 \prod_{i=1}^m (x - b_i)^{n_i} (x + b_i)^{n_i}.$$

相应地有

$$\begin{aligned} H(x^2) &= \frac{P_1(x^2)}{Q_1(x^2)} = \sum_{i=1}^m \sum_{k=1}^{n_i} \left[\frac{B_{ik}}{(x - b_i)^k} + \frac{B'_{ik}}{(x + b_i)^k} \right], \\ &= \sum_{i=1}^m \sum_{k=1}^{n_i} \left[\frac{A_{ik}}{(x - b_i)^k} + \frac{A'_{ik}}{(x + b_i)^k} \right]. \end{aligned}$$

由 $H(x^2) = H((-x)^2)$, 从 $H(x^2)$ 的分解式的唯一性, 比较系数, 即得 $A'_{ik} = A_{ik} (k = 1, 2, \dots, n_i; i = 1, 2, \dots, m)$. 最后得到

$$\begin{aligned} R(x) &= P(x^2) + H(x^2) = P(x^2) + \sum_{i=1}^m \sum_{k=1}^{n_i} \\ &\left[\frac{A_{ik}}{(b_i - x)^k} + \frac{A_{ik}}{(b_i + x)^k} \right], \text{ 其中 } b_i \text{ 为分母 } Q_1(x^2) \text{ 的根,} \\ &A_{ik} \text{ 为常数.} \end{aligned}$$

1925. 计算

$$\int \frac{dx}{1+x^{2n}},$$

式中 n 为正整数.

解 先将被积函数分解成部分分式之和, 我们可以证明

$$\frac{1}{1+x^{2n}} = \frac{1}{n} \sum_{k=1}^n \frac{1 - x \cos \frac{2k-1}{2n} \pi}{x^2 - 2x \cos \frac{2k-1}{2n} \pi + 1}.$$

事实上, 记多项式 $x^{2n} + 1$ 的 $2n$ 个根为 $a_k (k = 1, 2, \dots, 2n)$, 显然 $a_k = \cos \frac{2k-1}{2n} \pi + j \sin \frac{2k-1}{2n} \pi$, 其中 $j = \sqrt{-1}$ 为虚数单位.

于是, $|a_k| = 1, a_k^{2n} = -1, \bar{a}_k = a_{2n-k+1}$,

$$a_k \cdot \bar{a}_k = 1, a_k + \bar{a}_k = 2 \cos \frac{2k-1}{2n} \pi.$$

设
$$\frac{1}{1+x^{2n}} = \sum_{k=1}^{2n} \frac{A_k}{x - a_k}$$

即
$$1 = \sum_{k=1}^{2n} \frac{A_k(1+x^{2n})}{x - a_k}$$

令 $x \rightarrow a_i$ 并应用洛比塔法则, 即得

$$\begin{aligned} 1 &= \lim_{x \rightarrow a_i} \sum_{k=1}^{2n} \frac{A_k(1+x^{2n})}{x - a_k} = \lim_{x \rightarrow a_i} \frac{A_i(1+x^{2n})}{x - a_i} \\ &= \lim_{x \rightarrow a_i} (2n A_i x^{2n-1}) \\ &= 2n A_i \cdot \frac{a_i^{2n}}{a_i} = - \frac{2n A_i}{a_i} (i = 1, 2, \dots, 2n), \end{aligned}$$

即
$$A_k = - \frac{a_k}{2n} (k = 1, 2, \dots, 2n).$$

于是,

$$\begin{aligned}
\frac{1}{1+x^{2n}} &= -\frac{1}{2n} \sum_{k=1}^{2n} \frac{a_k}{x-a_k} \\
&= -\frac{1}{2n} \sum_{k=1}^n \left(\frac{a_k}{x-a_k} + \frac{\bar{a}_k}{x-\bar{a}_k} \right) \\
&= -\frac{1}{2n} \sum_{k=1}^n \frac{(a_k + \bar{a}_k)x - 2a_k\bar{a}_k}{x^2 - (a_k + \bar{a}_k)x + a_k \cdot \bar{a}_k} \\
&= \frac{1}{n} \sum_{k=1}^n \frac{1 - x \cos \frac{2k-1}{2n}\pi}{x^2 - 2x \cos \frac{2k-1}{2n}\pi + 1}.
\end{aligned}$$

最后得到

$$\begin{aligned}
\int \frac{dx}{1+x^{2n}} &= \frac{1}{n} \sum_{k=1}^n \int \frac{1 - x \cos \frac{2k-1}{2n}\pi}{x^2 - 2x \cos \frac{2k-1}{2n}\pi + 1} dx \\
&= -\frac{1}{2n} \sum_{k=1}^n \left[\cos \frac{2k-1}{2n}\pi \cdot \int \frac{2x - 2 \cos \frac{2k-1}{2n}\pi}{x^2 - 2x \cos \frac{2k-1}{2n}\pi + 1} dx \right. \\
&\quad \left. + \frac{1}{n} \sum_{k=1}^n \left(\sin^2 \frac{2k-1}{2n}\pi \cdot \int \frac{dx}{\left(x - \cos \frac{2k-1}{2n}\pi \right)^2 + \sin^2 \frac{2k-1}{2n}\pi} \right) \right] \\
&= -\frac{1}{2n} \sum_{k=1}^n \left[\cos \frac{2k-1}{2n}\pi \cdot \ln \left(x^2 - 2x \cos \frac{2k-1}{2n}\pi + 1 \right) \right]
\end{aligned}$$

$$+ \frac{1}{n} \sum_{k=1}^n \left[\sin \frac{2k-1}{2n} \pi \cdot \operatorname{arctg} \frac{x - \cos \frac{2k-1}{2n} \pi}{\sin \frac{2k-1}{2n} \pi} \right] + C.$$

§ 3. 无理函数的积分法

化被积函数为有理函数,以求下列积分:

1926. $\int \frac{dx}{1 + \sqrt{x}}.$

解 设 $\sqrt{x} = t$, 则 $x = t^2, dx = 2t dt$.

代入得

$$\begin{aligned} \int \frac{dx}{1 + \sqrt{x}} &= 2 \int \frac{t dt}{1 + t} = 2 \int \left(1 - \frac{1}{1 + t} \right) dt \\ &= 2(t - \ln(1 + t)) + C = 2\sqrt{x} - 2\ln(1 + \sqrt{x}) \\ &\quad + C. \end{aligned}$$

1927. $\int \frac{dx}{x(1 + 2\sqrt{x} + \sqrt[3]{x})}.$

解 设 $\sqrt[6]{x} = t$, 则 $x = t^6, dx = 6t^5 dt$.

代入得

$$\begin{aligned} &\int \frac{dx}{x(1 + 2\sqrt{x} + \sqrt[3]{x})} \\ &= 6 \int \frac{dt}{t(1 + 2t^3 + t^2)} \\ &= 6 \int \frac{dt}{t(1 + t)(2t^2 - t + 1)} \\ &= 6 \int \left(\frac{1}{t} - \frac{1}{4(1 + t)} - \frac{6t - 1}{4(2t^2 - t + 1)} \right) dt \end{aligned}$$

$$\begin{aligned}
&= 6 \left\{ \ln t - \frac{1}{4} \ln(1+t) - \frac{3}{8} \int \frac{4t-1}{2t^2-t+1} dt \right. \\
&\quad \left. - \frac{1}{16} \int \frac{d\left(t - \frac{1}{4}\right)}{\left(t - \frac{1}{4}\right)^2 + \frac{7}{16}} \right\} \\
&= 6 \left\{ \ln|t| - \frac{1}{4} \ln|1+t| - \frac{3}{8} \ln(2t^2-t+1) \right. \\
&\quad \left. - \frac{1}{4\sqrt{7}} \operatorname{arctg} \frac{4t-1}{\sqrt{7}} \right\} + C \\
&= \frac{3}{4} \ln \frac{t^3}{(1+t)^2(2t^2-t+1)^3} \\
&\quad - \frac{3}{2\sqrt{7}} \operatorname{arctg} \frac{4t-1}{\sqrt{7}} + C \\
&= \frac{3}{4} \ln \frac{x \cdot \sqrt[3]{x}}{(1+\sqrt[3]{x})^2(2\sqrt[3]{x}-\sqrt[6]{x}+1)^3} \\
&\quad - \frac{3}{2\sqrt{7}} \operatorname{arctg} \frac{4\sqrt[6]{x}-1}{\sqrt{7}} + C.
\end{aligned}$$

1928⁺. $\int \frac{x \sqrt[3]{2+x}}{x + \sqrt[3]{2+x}} dx.$

解 设 $\sqrt[3]{2+x} = t$, 则 $x = t^3 - 2, dx = 3t^2 dt$.

代入得

$$\begin{aligned}
&\int \frac{x \sqrt[3]{2+x}}{x + \sqrt[3]{2+x}} dx = 3 \int \frac{t^6 - 2t^3}{t^3 + t - 2} dt \\
&= 3 \int \left(t^3 - t + \frac{t^2 - 2t}{t^3 + t - 2} \right) dt \\
&= \frac{3}{4} t^4 - \frac{3}{2} t^2 + 3 \int \left[-\frac{1}{4(t-1)} + \frac{\frac{5}{4}t - \frac{1}{2}}{t^2 + t + 2} \right] dt
\end{aligned}$$

$$\begin{aligned}
&= 4 \int \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3} \right) dt \\
&= -\frac{4}{1+t} + \frac{2}{(1+t)^2} + C \\
&= \frac{2}{(1+\sqrt[4]{x})^2} - \frac{4}{1+\sqrt[4]{x}} + C.
\end{aligned}$$

1931. $\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx.$

解 设 $\sqrt{\frac{x+1}{x-1}} = t$, 则

$$x = \frac{t^2 + 1}{t^2 - 1}, dx = -\frac{4t}{(t^2 - 1)^2} dt.$$

代入得

$$\begin{aligned}
\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx &= \int \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} dx \\
&= -4 \int \frac{t dt}{(t-1)(t+1)^3} \\
&= \int \left(-\frac{2}{(t+1)^3} + \frac{1}{(t+1)^2} + \frac{1}{2(t+1)} \right. \\
&\quad \left. - \frac{1}{2(t-1)} \right) dt \\
&= \frac{1}{(t+1)^2} - \frac{1}{t+1} + \frac{1}{2} \ln \left| \frac{t+1}{t-1} \right| + C_1 \\
&= \frac{1}{2} x^2 - \frac{1}{2} x \sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C.
\end{aligned}$$

如果不限制将被积函数化为有理函数, 本题的解法可简单些. 事实上,

$$\begin{aligned}
& \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx \\
&= \int \frac{(\sqrt{x+1} - \sqrt{x-1})^2}{(x+1) - (x-1)} dx \\
&= \int (x - \sqrt{x^2-1}) dx \\
&= \frac{1}{2}x^2 - \frac{1}{2}x \sqrt{x^2-1} + \frac{1}{2} \ln|x + \sqrt{x^2-1}| + C.
\end{aligned}$$

1932. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}.$

解 设 $\sqrt[3]{\frac{x+1}{x-1}} = t$, 则

$$x = \frac{t^3+1}{t^3-1}, dx = -\frac{6t^2}{(t^3-1)^2} dt.$$

代入得

$$\begin{aligned}
\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} &= -\frac{3}{2} \int dt = -\frac{3}{2}t + C \\
&= -\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + C.
\end{aligned}$$

1933. $\int \frac{x dx}{\sqrt[4]{x^3(a-x)}} \quad (a > 0).$

解 设 $\sqrt[4]{\frac{a-x}{x}} = t$, 则 $x = \frac{a}{1+t^4}$,

$$dx = -\frac{4at^3}{(1+t^4)^2} dt.$$

代入得

$$\int \frac{x dx}{\sqrt[4]{x^3(a-x)}} = \int \frac{dx}{\sqrt[4]{\frac{a-x}{x}}}$$

$$\begin{aligned}
&= -4a \int \frac{t^2}{(1+t^4)^2} dt \\
&= -4a \int \left(\frac{t}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)} \right)^2 dt \\
&= -\frac{a}{2} \int \left(\frac{1}{t^2 - t\sqrt{2} + 1} - \frac{1}{t^2 + t\sqrt{2} + 1} \right)^2 dt \\
&= -\frac{a}{2} \int \frac{dt}{(t^2 - t\sqrt{2} + 1)^2} - \frac{a}{2} \int \frac{dt}{(t^2 + t\sqrt{2} + 1)^2} \\
&\quad + a \int \frac{dt}{t^4 + 1}.
\end{aligned}$$

现在分别求上述积分,利用 1921 题的递推公式,即得

$$\begin{aligned}
&\int \frac{dt}{(t^2 - t\sqrt{2} + 1)^2} = \frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} \\
&\quad + \int \frac{dt}{t^2 - t\sqrt{2} + 1} \\
&= \frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} + \int \frac{d\left(t - \frac{\sqrt{2}}{2}\right)}{\left(t - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \\
&= \frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} + \sqrt{2} \operatorname{arctg}(\sqrt{2}t - 1) + C_1
\end{aligned}$$

及

$$\begin{aligned}
&\int \frac{dt}{(t^2 + t\sqrt{2} + 1)^2} \\
&= \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} + \int \frac{dt}{t^2 + t\sqrt{2} + 1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} + \int \frac{d\left(t + \frac{\sqrt{2}}{2}\right)}{\left(t + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \\
&= \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} + \sqrt{2} \operatorname{arctg}(\sqrt{2}t + 1) \\
&\quad + C_2.
\end{aligned}$$

利用 1884 题的结果, 即得

$$\begin{aligned}
\int \frac{dt}{t^4 + 1} &= \frac{1}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\
&\quad + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t\sqrt{2}}{1 - t^2} + C_3.
\end{aligned}$$

最后得到

$$\begin{aligned}
\int \frac{x dx}{\sqrt[4]{x^3(a-x)}} &= -\frac{a}{2} \left[\frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} \right. \\
&\quad \left. + \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} \right] - \frac{a\sqrt{2}}{2} \{ \operatorname{arctg}(\sqrt{2}t - 1) \\
&\quad + \operatorname{arctg}(\sqrt{2}t + 1) \} + \frac{a}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\
&\quad + \frac{a}{2\sqrt{2}} \operatorname{arctg} \left(\frac{t\sqrt{2}}{1 - t^2} \right) + C_4 \\
&= -\frac{at^3}{1 + t^4} + \frac{a}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\
&\quad - \frac{a}{2\sqrt{2}} \operatorname{arctg} \left(\frac{t\sqrt{2}}{1 - t^2} \right) + C_4 \\
&= -\frac{at^3}{1 + t^4} + \frac{a}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1}
\end{aligned}$$

$$+ \frac{a}{2\sqrt{2}} \operatorname{arctg} \left(\frac{1-t^2}{t\sqrt{2}} \right) + C,$$

其中 $t = \sqrt[4]{\frac{a-x}{x}} \quad (0 < x < a).$

1934. $\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} \quad (n \text{ 为自然数}).$

解 当 $a=b$ 时, 显然被积函数为 $(x-a)^{-2}$, 因此积分

为 $-\frac{1}{x-a} + C$; 当 $a \neq b$ 时, 设 $\sqrt[n]{\frac{x-b}{x-a}} = t$, 则

$$x = a + \frac{a-b}{t^n - 1}, dx = -\frac{n(a-b)t^{n-1}}{(t^n - 1)^2} dt,$$

$$x-a = \frac{a-b}{t^n - 1}, x-b = \frac{(a-b)t^n}{t^n - 1},$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} &= -\frac{n}{a-b} \int dt \\ &= -\frac{n}{a-b} t + C \end{aligned}$$

$$= -\frac{n}{a-b} \sqrt[n]{\frac{x-b}{x-a}} + C.$$

1935. $\int \frac{dx}{1 + \sqrt{x} + \sqrt{1+x}}.$

解 设 $\sqrt{x} = \frac{t^2-1}{2t}$ 并限制 $t > 1$, 则

$$x = \left(\frac{t^2-1}{2t} \right)^2, dx = \frac{t^4-1}{2t^3} dt, \sqrt{x+1} = \frac{t^2+1}{2t},$$

$$t = \sqrt{x} + \sqrt{x+1}.$$

代入得

$$\begin{aligned}
& \int \frac{dx'}{1 + \sqrt{x} + \sqrt{x+1}} = \frac{1}{2} \int \frac{t^4 - 1}{t^3(t+1)} dt \\
&= \frac{1}{2} \int \left(1 - \frac{1}{t} + \frac{1}{t^2} - \frac{1}{t^3} \right) dt \\
&= \frac{1}{2} \left(t - \ln t - \frac{1}{t} + \frac{1}{2t^2} \right) + C_1 \\
&= \sqrt{x} - \frac{1}{2} \ln(\sqrt{x} + \sqrt{x+1}) \\
&\quad + \frac{x}{2} - \frac{1}{2} \sqrt{x(x+1)} + C.
\end{aligned}$$

1936. 证明: 若

$$p + q = kn,$$

式中 k 为整数, 则积分

$$\int R[x, (x-a)^{\frac{p}{n}}(x-b)^{\frac{q}{n}}] dx$$

(式中 R 为有理函数及 p, q, n 为整数) 为初等函数.

证 当 $a = b$ 时, $(x-a)^{\frac{p}{n}}(x-b)^{\frac{q}{n}} = (x-a)^k$, 则积分显然为初等函数.

当 $a \neq b$ 时, 设 $\frac{x-a}{x-b} = y (\neq 1)$, 则

$$x = \frac{a-by}{1-y}, dx = \frac{a-b}{(1-y)^2} dy,$$

$$x-a = \frac{(a-b)y}{1-y}, x-b = \frac{a-b}{1-y}.$$

代入得

$$\begin{aligned}
& \int R[x, (x-a)^{\frac{p}{n}}(x-b)^{\frac{q}{n}}] dx \\
&= (a-b) \int R \left[\frac{a-by}{1-y}, y^{\frac{p}{n}} \left(\frac{a-b}{1-y} \right)^k \right] \frac{dy}{(1-y)^2}.
\end{aligned}$$

再设 $\sqrt[n]{y} = t$, 则 $y = t^n$, $dy = nt^{n-1}dt$. 从而上述积分化为

$$\begin{aligned} & \int R[x, (x-a)^{\frac{k}{n}}(x-b)^{\frac{l}{n}}]dx \\ &= n(a-b) \int R\left[\frac{a-bt^n}{1-t^n}, t^p\left(\frac{a-b}{1-t^n}\right)^k\right] \frac{t^n-1}{(1-t^n)^2}dt, \end{aligned}$$

因为被积函数为 t 的有理函数, 所以积分是初等函数, 求最简单二次无理式的积分:

1937. $\int \frac{x^2}{\sqrt{1+x+x^2}}dx.$

$$\begin{aligned} \text{解} \quad & \int \frac{x^2}{\sqrt{1+x+x^2}}dx = \int \frac{x^2+x+1}{\sqrt{x^2+x+1}}dx \\ & - \frac{1}{2} \int \frac{2x+1}{\sqrt{1+x+x^2}}dx - \frac{1}{2} \int \frac{dx}{\sqrt{1+x+x^2}} \\ &= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x+\frac{1}{2}\right) \\ & - \frac{1}{2} \int \frac{d(1+x+x^2)}{\sqrt{1+x+x^2}} - \frac{1}{2} \int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} \\ &= \frac{2x+1}{4} \sqrt{1+x+x^2} + \frac{3}{8} \ln\left(x+\frac{1}{2} + \sqrt{1+x+x^2}\right) \\ & - \sqrt{1+x+x^2} - \frac{1}{2} \ln\left(x+\frac{1}{2} + \sqrt{1+x+x^2}\right) + C \\ &= \frac{2x-3}{4} \sqrt{1+x+x^2} - \frac{1}{8} \ln\left(x+\frac{1}{2} + \sqrt{1+x+x^2}\right) + C. \end{aligned}$$

$$1938^+. \int \frac{dx}{(1+x)\sqrt{x^2+x+1}}.$$

解 设 $x+1=\frac{1}{t}$, 则

$$x=\frac{1-t}{t}, dx=-\frac{1}{t^2}dt,$$

$$\sqrt{x^2+x+1}=\frac{\sqrt{t^2-t+1}}{|t|}=\operatorname{sgnt} \cdot \frac{\sqrt{t^2-t+1}}{t}.$$

代入得

$$\begin{aligned} & \int \frac{dx}{(1+x)\sqrt{x^2+x+1}} \\ &= -\operatorname{sgnt} \int \frac{dt}{\sqrt{t^2-t+1}} \\ &= -\operatorname{sgnt} \cdot \ln \left| t - \frac{1}{2} + \sqrt{t^2-t+1} \right| + C_1 \\ &= -\operatorname{sgn}(x+1) \\ & \quad \cdot \ln \left| \frac{1-x+2[\operatorname{sgn}(x+1)] \cdot \sqrt{x^2+x+1}}{2(x+1)} \right| \\ & \quad + C_1. \end{aligned}$$

当 $x+1>0$ 时,

$$\begin{aligned} & \int \frac{dx}{(1+x)\sqrt{x^2+x+1}} \\ &= -\ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C_1 \end{aligned}$$

当 $x+1<0$ 时,

$$\begin{aligned} & \int \frac{dx}{(1+x)\sqrt{x^2+x+1}} \\ &= \ln \left| \frac{1-x-2\sqrt{x^2+x+1}}{2(1+x)} \right| + C_1 \end{aligned}$$

$$= \ln \left| \frac{-3(x+1)}{2(1-x+2\sqrt{x^2+x+1})} \right| + C_1$$

$$= -\ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C.$$

总之,

$$\begin{aligned} & \int \frac{dx}{(1+x)\sqrt{x^2+x+1}} \\ &= -\ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C, \end{aligned}$$

1939. $\int \frac{dx}{(1-x)^2 \sqrt{1-x^2}}.$

解 设 $\sqrt{\frac{1-x}{1+x}} = t$, 则

$$x = \frac{1-t^2}{1+t^2}, dx = -\frac{4t}{(1+t^2)^2} dt,$$

$$1-x = \frac{2t^2}{1+t^2}, \sqrt{1-x^2} = \frac{2t}{1+t^2}.$$

代入得

$$\begin{aligned} & \int \frac{dx}{(1-x)^2 \sqrt{1-x^2}} = -\frac{1}{2} \int \frac{1+t^2}{t^4} dt \\ &= \frac{1}{6t^3} + \frac{1}{2t} + C \\ &= \frac{2-x}{3(1-x)^2} \sqrt{1-x^2} + C. \end{aligned}$$

1940. $\int \frac{\sqrt{x^2+2x+2}}{x} dx.$

解 设 $\sqrt{x^2+2x+2} = t-x$, 则

$$x = \frac{t^2-2}{2(t+1)}, dx = \frac{t^2+2t+2}{2(t+1)^2} dt,$$

$$= \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) + [\operatorname{sgn}(1+x)] \\ \cdot \ln\left|\frac{3+x+2[\operatorname{sgn}(x+1)]\sqrt{1-x-x^2}}{2(1+x)}\right| + C_1.$$

当 $x+1>0$ 时,

$$\int \frac{xdx}{(1+x)\sqrt{1-x-x^2}} = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) \\ + \ln\left|\frac{3+x+2\sqrt{1-x-x^2}}{1+x}\right| + C;$$

当 $x+1<0$ 时,

$$\int \frac{xdx}{(1+x)\sqrt{1-x-x^2}} = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) \\ - \ln\left|\frac{3+x-2\sqrt{1-x-x^2}}{2(1+x)}\right| + C_1 \\ = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) + \ln\left|\frac{3+x+2\sqrt{1-x-x^2}}{1+x}\right| + C.$$

总之,

$$\int \frac{xdx}{(1+x)\sqrt{1-x-x^2}} = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) \\ + \ln\left|\frac{3+x+2\sqrt{1-x-x^2}}{1+x}\right| + C.$$

以后诸题中,出现二次无理式时也会碰到用 sgnt 的问题,可参照 1938 题及 1941 题类似地处理.在解这类习题时,不妨就开方后取正值求解.如无特殊情况,今后不再另加说明.

1942. $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}} dx.$

$$\begin{aligned}
\text{解} \quad & \int \frac{1-x+x^2}{\sqrt{1+x-x^2}} dx = \int \frac{(x^2-x-1)+2}{\sqrt{1+x-x^2}} dx \\
& = - \int \sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2} d\left(x - \frac{1}{2}\right) \\
& \quad + 2 \int \frac{d\left(x - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2}} \\
& = \frac{1-2x}{4} \sqrt{1+x-x^2} - \frac{5}{8} \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) \\
& \quad + 2 \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C \\
& = \frac{1-2x}{4} \sqrt{1+x-x^2} - \frac{11}{8} \arcsin\left(\frac{1-2x}{\sqrt{5}}\right) + C.
\end{aligned}$$

利用公式

$$\int \frac{P_n(x)}{y} dx = Q_{n-1}(x)y + \lambda \int \frac{dx}{y},$$

式中 $y = \sqrt{ax^2+bx+c}$, $P_n(x)$ 为 n 次多项式, $Q_{n-1}(x)$ 为 $n-1$ 次多项式及 λ 为常数, 求下列积分:

1943. $\int \frac{x^3}{\sqrt{1+2x-x^2}} dx.$

$$\begin{aligned}
\text{解} \quad & \text{设} \int \frac{x^3}{\sqrt{1+2x-x^2}} dx \\
& = (ax^2+bx+c) \sqrt{1+2x-x^2} \\
& \quad + \lambda \int \frac{dx}{\sqrt{1+2x-x^2}},
\end{aligned}$$

两边对 x 求导数, 得

$$\frac{x^3}{\sqrt{1+2x-x^2}} = (2ax+b) \sqrt{1+2x-x^2}$$

$$+ \frac{(ax^2 + bx + c)(1-x)}{\sqrt{1+2x-x^3}} + \frac{\lambda}{\sqrt{1+2x-x^2}}.$$

从而有

$$x^3 = (2ax + b)(1 + 2x - x^2) + (ax^2 + bx + c) \cdot (1 - x) + \lambda.$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^3 & -3a = 1, \\ x^2 & 5a - 2b = 0, \\ x^1 & 2a + 3b - c = 0, \\ x^0 & b + c + \lambda = 0. \end{array}$$

由此, $a = -\frac{1}{3}$, $b = -\frac{5}{6}$, $c = -\frac{19}{6}$, $\lambda = 4$.

于是,

$$\begin{aligned} \int \frac{x^3}{\sqrt{1+2x-x^2}} dx &= -\frac{19+5x+2x^2}{6} \sqrt{1+2x-x^2} \\ &+ 4 \int \frac{dx}{\sqrt{1+2x-x^2}} \\ &= -\frac{19+5x+2x^2}{6} \sqrt{1+2x-x^2} \\ &+ 4 \arcsin\left(\frac{x-1}{\sqrt{2}}\right) + C. \end{aligned}$$

1944. $\int \frac{x^{10}}{\sqrt{1+x^2}} dx.$

解 设 $\int \frac{x^{10}}{\sqrt{1+x^2}} dx = (ax^9 + bx^8 + cx^7 + dx^6 + ex^5 + fx^4 + gx^3 + hx^2 + lx + m) \sqrt{1+x^2} + \lambda \int \frac{dx}{\sqrt{1+x^2}},$

从而有

$$\begin{aligned}x^{10} &\equiv (9ax^8 + 8bx^7 + 7cx^6 + 6dx^5 + 5ex^4 + 4fx^3 + 3gx^2 \\&\quad + 2hx + l)(1+x^2) \\&\quad + x(ax^9 + bx^8 + cx^7 + dx^6 + ex^5 + fx^4 + gx^3 \\&\quad + hx^2 + lx + m) + \lambda.\end{aligned}$$

比较等式两端 x 的同次幂系数, 求得

$$a = \frac{1}{10}, b = 0, c = -\frac{9}{80}, d = 0,$$

$$e = \frac{21}{160}, f = 0, g = -\frac{21}{128}, h = 0,$$

$$l = \frac{63}{256}, m = 0, \lambda = -\frac{63}{256}.$$

于是,

$$\begin{aligned}\int \frac{x^{10}}{\sqrt{1+x^2}} dx &= \left(\frac{63}{256}x - \frac{21}{128}x^3 + \frac{21}{160}x^5 - \frac{9}{80}x^7 \right. \\&\quad \left. + \frac{1}{10}x^9 \right) \sqrt{1+x^2} - \frac{63}{256} \ln(x + \sqrt{1+x^2}) + C.\end{aligned}$$

1945. $\int x^4 \sqrt{a^2 - x^2} dx.$

$$\begin{aligned}\text{解} \quad \int x^4 \sqrt{a^2 - x^2} dx &= \int \frac{x^4(a^2 - x^2)}{\sqrt{a^2 - x^2}} dx \\&= (Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F) \sqrt{a^2 - x^2} \\&\quad + \lambda \int \frac{dx}{\sqrt{a^2 - x^2}},\end{aligned}$$

从而有

$$\begin{aligned}x^4(a^2 - x^2) &\equiv (5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx \\&\quad + E)(a^2 - x^2) - x(Ax^5 + Bx^4 + Cx^3 + Dx^2 \\&\quad + Ex + F) + \lambda.\end{aligned}$$

比较等式两端 x 的同次幂系数,求得

$$A = \frac{1}{6}, B = 0, C = -\frac{a^2}{24}, D = 0,$$

$$E = -\frac{a^4}{16}, F = 0, \lambda = \frac{a^4}{16}.$$

于是,

$$\int x^4 \sqrt{a^2 - x^2} dx = \left(\frac{1}{6} x^5 - \frac{a^2}{24} x^3 - \frac{a^4}{16} x \right) \sqrt{a^2 - x^2} + \frac{a^4}{16} \arcsin \frac{x}{|a|} + C \quad (a \neq 0).$$

1946. $\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx.$

解 设 $\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx$

$$= (ax^2 + bx + c) \sqrt{x^2 + 4x + 3}$$

$$+ \lambda \int \frac{dx}{\sqrt{x^2 + 4x + 3}},$$

从而有

$$x^3 - 6x^2 + 11x - 6 \equiv (2ax + b)(x^2 + 4x + 3) + (x + 2)(ax^2 + bx + c) + \lambda.$$

比较等式两端 x 的同次幂系数,求得

$$a = \frac{1}{3}, b = -\frac{14}{3}, c = 37, \lambda = -66.$$

于是,

$$\begin{aligned} & \int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx \\ &= \left(\frac{1}{3} x^2 - \frac{14}{3} x + 37 \right) \sqrt{x^2 + 4x + 3} \\ & \quad - 66 \ln |x + 2 + \sqrt{x^2 + 4x + 3}| + C. \end{aligned}$$

1947. $\int \frac{dx}{x^3 \sqrt{x^2+1}}.$

解 设 $x = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$, 这里碰到二次无理式

$\sqrt{x^2+1}$ 需引用 sgnt 的问题, 不妨设

$$\sqrt{x^2+1} = \frac{\sqrt{t^2+1}}{t} \quad (t > 0).$$

代入得

$$\begin{aligned} \int \frac{dx}{x^3 \sqrt{x^2+1}} &= - \int \frac{t^2}{\sqrt{t^2+1}} dt \\ &= - \int \frac{(t^2+1) - 1}{\sqrt{t^2+1}} dt \\ &= - \int \sqrt{t^2+1} dt + \int \frac{dt}{\sqrt{t^2+1}} \\ &= -\frac{t}{2} \sqrt{t^2+1} - \frac{1}{2} \ln |t + \sqrt{t^2+1}| \\ &\quad + \ln |t + \sqrt{t^2+1}| + C \\ &= -\frac{\sqrt{x^2+1}}{2x^2} + \frac{1}{2} \ln \frac{1 + \sqrt{x^2+1}}{|x|} + C. \end{aligned}$$

1948⁺. $\int \frac{dx}{x^4 \sqrt{x^2-1}}.$

解 不妨设 $x = \frac{1}{t} > 0$, 则 $dx = -\frac{1}{t^2} dt$. 由 $|x| > 1$ 知必有 $|t| < 1$, 则有

$$\sqrt{x^2-1} = \frac{\sqrt{1-t^2}}{t} \quad (0 < t < 1).$$

代入得

$$\int \frac{dx}{x^4 \sqrt{x^2-1}} = - \int \frac{t^3}{\sqrt{1-t^2}} dt$$

$$\begin{aligned}
&= \int \frac{t(1-t^2)-t}{\sqrt{1-t^2}} dt \\
&= \int t \sqrt{1-t^2} dt - \int \frac{t}{\sqrt{1-t^2}} dt \\
&= -\frac{1}{2} \int (1-t^2)^{\frac{1}{2}} d(1-t^2) \\
&\quad + \frac{1}{2} \int (1-t^2)^{-\frac{1}{2}} d(1-t^2) \\
&= -\frac{1}{3} (1-t^2)^{\frac{3}{2}} + (1-t^2)^{\frac{1}{2}} + C \\
&= \frac{1+2x^2}{3x^3} \sqrt{x^2-1} + C.
\end{aligned}$$

1949⁺. $\int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}}.$

解 设 $x-1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$. 不妨设 $t > 0$, 则有

$$\sqrt{x^2+3x+1} = \frac{\sqrt{5t^2+5t+1}}{t}.$$

代入得

$$\begin{aligned}
\int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}} &= - \int \frac{t^2}{\sqrt{5t^2+5t+1}} dt \\
&= (at+b) \sqrt{5t^2+5t+1} + \lambda \int \frac{dt}{\sqrt{5t^2+5t+1}},
\end{aligned}$$

从而

$$-t^2 \equiv a(1+5t+5t^2) + (5t + \frac{5}{2})(at+b) + \lambda.$$

比较等式两端 t 的同次幂系数, 求得

$$a = -\frac{1}{10}, b = \frac{3}{20}, \lambda = -\frac{11}{40}.$$

于是,

$$\begin{aligned}
 & \int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}} \\
 &= \left(-\frac{t}{10} + \frac{3}{20} \right) \sqrt{5t^2+5t+1} \\
 &\quad - \frac{11}{40} \int \frac{dt}{\sqrt{5t^2+5t+1}} \\
 &= \frac{3-2t}{20} \sqrt{5t^2+5t+1} - \frac{11}{40\sqrt{5}} \ln \left| t + \frac{1}{2} \right. \\
 &\quad \left. + \sqrt{t^2+t+\frac{1}{5}} \right| + C_1 \\
 &= \frac{3x-5}{20(x-1)^2} \sqrt{x^2+3x+1} \\
 &\quad - \frac{11}{40\sqrt{5}} \ln \left| \frac{\sqrt{5}(x+1)+2\sqrt{x^2+3x+1}}{x-1} \right| \\
 &\quad + C.
 \end{aligned}$$

1950⁺. $\int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}}$

解 设 $x+1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2}dt$. 先设 $t > 0$, 则有

$$\sqrt{x^2+2x} = \frac{\sqrt{1-t^2}}{t}.$$

代入得

$$\begin{aligned}
 \int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}} &= - \int \frac{t^4}{\sqrt{1-t^2}} dt \\
 &= (at^3 + bt^2 + ct + e) \sqrt{1-t^2} + \lambda \int \frac{dt}{\sqrt{1-t^2}}.
 \end{aligned}$$

从而有

$$-t^4 \equiv (3at^2 + 2bt + c)(1 - t^2) - t(at^3 + bt^2 + ct + e) + \lambda.$$

比较等式两端 t 的同次幂系数, 求得

$$a = \frac{1}{4}, b = 0, c = \frac{3}{8}, e = 0, \lambda = -\frac{3}{8}.$$

于是,

$$\begin{aligned} \int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}} &= \left(\frac{1}{4}t^3 + \frac{3}{8}t \right) \\ &\cdot \sqrt{1-t^2} - \frac{3}{8} \int \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{3x^2+6x+5}{8(x+1)^4} \sqrt{x^2+2x} - \frac{3}{8} \arcsin \frac{1}{x+1} + C. \end{aligned}$$

再设 $t < 0$, 则答案前一项不改变符号, 但后一项要改变符号. 因此, 最后得到

$$\begin{aligned} \int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}} &= \frac{3x^2+6x+5}{8(x+1)^4} \sqrt{x^2+2x} \\ &- \frac{3}{8} \arcsin \frac{1}{|x+1|} + C, \end{aligned}$$

其中 $x > 0$ 或 $x < -2$.

1951. 在什么条件下, 积分

$$\int \frac{a_1x^2 + b_1x + c_1}{\sqrt{ax^2 + bx + c}} dx$$

是代数函数?

解 设 $\int \frac{a_1x^2 + b_1x + c_1}{\sqrt{ax^2 + bx + c}} dx$

$$= (Ax + B) \sqrt{ax^2 + bx + c} + \lambda \int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

从而有

$$a_1x^2 + b_1x + c_1 \equiv A(ax^2 + bx + c)$$

$$\begin{aligned}
&= - \int \frac{t dt}{\sqrt{2t^2 - 1}} - \int \frac{dt}{\sqrt{2t^2 - 1}} \\
&= - \frac{1}{2} \sqrt{2t^2 - 1} \\
&\quad - \frac{1}{\sqrt{2}} \ln \left| \sqrt{2} t + \sqrt{2t^2 - 1} \right| + C \\
&= \frac{\sqrt{1 + 2x - x^2}}{2(1 - x)} \\
&\quad - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sqrt{1 + 2x - x^2}}{1 - x} \right| + C.
\end{aligned}$$

1953. $\int \frac{x dx}{(x^2 - 1) \sqrt{x^2 - x - 1}}.$

解 $\int \frac{x dx}{(x^2 - 1) \sqrt{x^2 - x - 1}}$

$$\begin{aligned}
&= \frac{1}{2} \int \left(\frac{1}{x+1} + \frac{1}{x-1} \right) \frac{dx}{\sqrt{x^2 - x - 1}} \\
&= \frac{1}{2} \int \frac{dx}{(x+1) \sqrt{x^2 - x - 1}} \\
&\quad + \frac{1}{2} \int \frac{dx}{(x-1) \sqrt{x^2 - x - 1}} \\
&= \frac{1}{2} I_1 + \frac{1}{2} I_2.
\end{aligned}$$

对于 I_1 , 设 $x + 1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$. 不妨设 $t > 0$, 则有

$$\sqrt{x^2 - x - 1} = \frac{\sqrt{t^3 - 3t + 1}}{t}.$$

代入 I_1 , 得

$$I_1 = \int \frac{dx}{(x+1) \sqrt{x^2 - x - 1}} = - \int \frac{dt}{\sqrt{t^3 - 3t + 1}}$$

$$= -\ln\left|t - \frac{3}{2} + \sqrt{t^2 - 3t + 1}\right| + C_1$$

$$= -\ln\left|\frac{3x+1-2\sqrt{x^2-x-1}}{x+1}\right| + C_2;$$

对于 I_2 , 设 $x-1 = \frac{1}{t}$, 同上可得

$$\begin{aligned} I_2 &= \int \frac{dx}{(x-1)\sqrt{x^2-x-1}} \\ &= \arcsin\left(\frac{x-3}{|x-1|\sqrt{5}}\right) + C_3. \end{aligned}$$

于是,

$$\begin{aligned} &\int \frac{dx}{(x^2-1)\sqrt{x^2-x-1}} \\ &= -\frac{1}{2}\ln\left|\frac{3x+1-2\sqrt{x^2-x-1}}{x+1}\right| \\ &\quad + \frac{1}{2}\arcsin\left(\frac{x-3}{|x-1|\sqrt{5}}\right) + C. \end{aligned}$$

1954. $\int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx.$

$$\begin{aligned} \text{解} \quad &\int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx \\ &= \int \frac{x^2+x+1}{(x+1)^2} \cdot \frac{dx}{\sqrt{x^2+x+1}} \\ &= \int \frac{(x+1)^2 - (x+1) + 1}{(x+1)^2} \cdot \frac{dx}{\sqrt{x^2+x+1}} \\ &= \int \frac{dx}{\sqrt{x^2+x+1}} - \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} \\ &\quad + \int \frac{dx}{(x+1)^2\sqrt{x^2+x+1}} = I_1 - I_2 + I_3. \end{aligned}$$

对于 I_1 , 显然有

$$I_1 = \int \frac{dx}{\sqrt{x^2 + x + 1}} = \ln\left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C_1;$$

对于 I_2 , 利用 1938 题的结果, 即得

$$\begin{aligned} I_2 &= \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} \\ &= -\ln\left|\frac{1-x+2\sqrt{x^2+x+1}}{x+1}\right| + C_2; \end{aligned}$$

对于 I_3 , 设 $x+1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2}dt$. 不妨设 $t > 0$, 则有

$$\sqrt{x^2+x+1} = \frac{\sqrt{t^2-t+1}}{t}.$$

代入得

$$\begin{aligned} I_3 &= -\int \frac{tdt}{\sqrt{t^2-t+1}} \\ &= -\frac{1}{2}\int \frac{(2t-1)dt}{\sqrt{t^2-t+1}} - \frac{1}{2}\int \frac{dt}{\sqrt{t^2-t+1}} \\ &= -\sqrt{t^2-t+1} - \frac{1}{2}\ln\left|t - \frac{1}{2} + \sqrt{t^2-t+1}\right| \\ &\quad + C_3 \\ &= -\frac{\sqrt{x^2+x+1}}{x+1} \\ &\quad - \frac{1}{2}\ln\left|\frac{1-x+2\sqrt{x^2+x+1}}{x+1}\right| + C_4. \end{aligned}$$

于是, 最后得到

$$\int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx$$

$$= \ln\left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) - \frac{\sqrt{x^2 + x + 1}}{x + 1} \\ + \frac{1}{2} \ln\left|\frac{1 - x + 2\sqrt{x^2 + x + 1}}{x + 1}\right| + C.$$

如用下述解法更简单些:

$$\int \frac{\sqrt{x^2 + x + 1}}{(x + 1)^2} dx \\ = - \int \sqrt{x^2 + x + 1} d\left(\frac{1}{x + 1}\right) \\ = - \frac{\sqrt{x^2 + x + 1}}{x + 1} + \int \frac{\left(x + \frac{1}{2}\right) dx}{(x + 1) \sqrt{x^2 + x + 1}} \\ = - \frac{\sqrt{x^2 + x + 1}}{x + 1} + \int \frac{dx}{\sqrt{x^2 + x + 1}} \\ - \frac{1}{2} \int \frac{dx}{(x + 1) \sqrt{x^2 + x + 1}} \\ = - \frac{\sqrt{x^2 + x + 1}}{x + 1} \\ + \ln\left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) \\ + \frac{1}{2} \ln\left|\frac{1 - x + 2\sqrt{x^2 + x + 1}}{x + 1}\right| + C.$$

*) 利用 1938 题的结果.

1955. $\int \frac{x^3}{(1 + x) \sqrt{1 + 2x - x^2}} dx.$

解 $\int \frac{x^3}{(1 + x) \sqrt{1 + 2x - x^2}} dx \\ = \int \frac{(x^3 + 1) - 1}{(1 + x) \sqrt{1 + 2x - x^2}} dx$

$$\begin{aligned}
&= \int \frac{x^2 - x + 1}{\sqrt{1 + 2x - x^2}} dx - \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} \\
&= - \int \frac{1+2x-x^2}{\sqrt{1+2x-x^2}} dx + \frac{1}{2} \int \frac{(2x-2)dx}{\sqrt{1+2x-x^2}} \\
&\quad + 3 \int \frac{dx}{\sqrt{1+2x-x^2}} - \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}} \\
&= - \int \sqrt{2-(x-1)^2} d(x-1) \\
&\quad - \frac{1}{2} \int \frac{d(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 3 \int \frac{d(x-1)}{\sqrt{2-(x-1)^2}} - I_1 \\
&= \frac{1-x}{2} \sqrt{1+2x-x^2} - \arcsin\left(\frac{x-1}{\sqrt{2}}\right) \\
&\quad - \sqrt{1+2x-x^2} + 3 \arcsin\left(\frac{x-1}{\sqrt{2}}\right) - I_1 \\
&= -\frac{x+1}{2} \sqrt{1+2x-x^2} + 2 \arcsin\left(\frac{x-1}{\sqrt{2}}\right) - I_1.
\end{aligned}$$

对于 I_1 , 设 $x+1 = \frac{1}{t}$, 可得

$$\begin{aligned}
I_1 &= \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}} \\
&= \frac{1}{\sqrt{2}} \arcsin\left(\frac{x\sqrt{2}}{x+1}\right) + C_1.
\end{aligned}$$

于是, 最后得到

$$\begin{aligned}
&\int \frac{x^3}{(x+1)\sqrt{1+2x-x^2}} dx \\
&= -\frac{1+x}{2} \sqrt{1+2x-x^2} - 2 \arcsin\left(\frac{1-x}{\sqrt{2}}\right) \\
&\quad - \frac{1}{\sqrt{2}} \arcsin\left(\frac{x\sqrt{2}}{1+x}\right) + C.
\end{aligned}$$

1956. $\int \frac{xdx}{(x^2 - 3x + 2) \sqrt{x^2 - 4x + 3}}.$

解
$$\begin{aligned} & \int \frac{xdx}{(x^2 - 3x + 2) \sqrt{x^2 - 4x + 3}} \\ &= \int \left(\frac{2}{x-2} - \frac{1}{x-1} \right) \cdot \frac{dx}{\sqrt{x^2 - 4x + 3}} \\ &= \int \frac{2dx}{(x-2) \sqrt{x^2 - 4x + 3}} \\ &\quad - \int \frac{dx}{(x-1) \sqrt{x^2 - 4x + 3}} \\ &= 2I_1 - I_2. \end{aligned}$$

对于 I_1 , 设 $x - 2 = \frac{1}{t}$, 可得

$$\begin{aligned} I_1 &= \int \frac{dx}{(x-2) \sqrt{x^2 - 4x + 3}} \\ &= -\arcsin\left(\frac{1}{|x-2|}\right) + C_1; \end{aligned}$$

对于 I_2 , 设 $x - 1 = \frac{1}{t}$, 可得

$$\begin{aligned} I_2 &= \int \frac{dx}{(x-1) \sqrt{x^2 - 4x + 3}} \\ &= \frac{\sqrt{x^2 - 4x + 3}}{x-1} + C_2. \end{aligned}$$

于是, 最后得到

$$\begin{aligned} & \int \frac{xdx}{(x^2 - 3x + 2) \sqrt{x^2 - 4x + 3}} \\ &= -2\arcsin\left(\frac{1}{|x-2|}\right) - \frac{\sqrt{x^2 - 4x + 3}}{x-1} + C, \end{aligned}$$

其中 $x < 1$ 或 $x > 3$.

1957. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}.$

解 设 $x = \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$dx = \cos t dt, \sqrt{1-x^2} = \cos t.$$

代入得

$$\begin{aligned} \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} &= \int \frac{dt}{1+\sin^2 t} \\ &= \int \frac{dt}{2\sin^2 t + \cos^2 t} = \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2} \operatorname{tg} t)}{(\sqrt{2} \operatorname{tg} t)^2 + 1} \\ &= \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg}(\sqrt{2} \operatorname{tg} t) + C \\ &= \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left[\frac{x \sqrt{2}}{\sqrt{1-x^2}} \right] + C. \end{aligned}$$

1958. $\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}.$

解 当 $x > 1$ 时, 设 $x = \sec t$, 并限制 $0 < t < \frac{\pi}{2}$, 则

$$dx = \sec t \cdot \operatorname{tg} t dt, \sqrt{x^2-1} = \operatorname{tg} t.$$

代入得

$$\begin{aligned} \int \frac{dx}{(x^2+1)\sqrt{x^2-1}} &= \int \frac{\sec t dt}{1+\sec^2 t} \\ &= \int \frac{\cos t}{\cos^2 t + 1} dt = \int \frac{d(\sin t)}{2 - \sin^2 t} \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sin t}{\sqrt{2} - \sin t} \right| + C \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}x + \sqrt{x^2-1}}{\sqrt{2}x - \sqrt{x^2-1}} \right| + C. \end{aligned}$$

当 $x < -1$ 时, 仍设 $x = \sec t$, 但限制 $\pi < t < \frac{3}{2}\pi$,
经计算可获得同样的结果.

总之, 当 $|x| > 1$ 时,

$$\int \frac{dx}{(x^2 + 1) \sqrt{x^2 - 1}} = \frac{1}{2\sqrt{2}} \ln \left| \frac{x\sqrt{2} + \sqrt{x^2 - 1}}{x\sqrt{2} - \sqrt{x^2 - 1}} \right| + C.$$

1959. $\int \frac{dx}{(1 - x^4) \sqrt{1 + x^2}}.$

解 设 $x = \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$ 且 $|t| \neq \frac{\pi}{4}$, 则

$$dx = \sec^2 t dt, \sqrt{1 + x^2} = \sec t.$$

代入得

$$\begin{aligned} \int \frac{dx}{(1 - x^4) \sqrt{x^2 + 1}} &= \int \frac{\sec^2 t dt}{(1 - \operatorname{tg}^4 t) \sec t} \\ &= \int \frac{\cos^3 t dt}{1 - 2\sin^2 t} = \int \frac{1 - \sin^2 t}{1 - 2\sin^2 t} d(\sin t) \\ &= \frac{1}{2} \int \frac{1 - 2\sin^2 t}{1 - 2\sin^2 t} d(\sin t) + \frac{1}{2} \int \frac{d(\sin t)}{1 - 2\sin^2 t} \\ &= \frac{1}{2} \sin t + \frac{1}{4\sqrt{2}} \ln \left| \frac{1 + \sqrt{2} \sin t}{1 - \sqrt{2} \sin t} \right| + C \\ &= \frac{x}{2\sqrt{1 + x^2}} + \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{1 + x^2} + x\sqrt{2}}{\sqrt{1 + x^2} - x\sqrt{2}} \right| \\ &\quad + C \quad (|x| \neq 1). \end{aligned}$$

1960. $\int \frac{\sqrt{x^2 + 2}}{x^2 + 1} dx.$

解 $\int \frac{\sqrt{x^2 + 2}}{x^2 + 1} dx = \int \frac{(x^2 + 2) dx}{(x^2 + 1) \sqrt{x^2 + 2}}$

$$\begin{aligned}
&= \int \left(1 + \frac{1}{x^2 + 1} \right) \cdot \frac{dx}{\sqrt{x^2 + 2}} \\
&= \int \frac{dx}{\sqrt{x^2 + 2}} + \int \frac{dx}{(x^2 + 1) \sqrt{x^2 + 2}} \\
&= \ln(x + \sqrt{x^2 + 2}) + I_1.
\end{aligned}$$

对于 I_1 , 设 $x = \sqrt{2} \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$dx = \sqrt{2} \sec^2 t dt, \quad \sqrt{x^2 + 2} = \sqrt{2} \sec t.$$

代入得

$$\begin{aligned}
I_1 &= \int \frac{dx}{(x^2 + 1) \sqrt{x^2 + 2}} = \int \frac{\sec t dt}{1 + 2 \operatorname{tg}^2 t} \\
&= \int \frac{\cos t dt}{1 + \sin^2 t} = \int \frac{d(\sin t)}{1 + \sin^2 t} = \operatorname{arctg}(\sin t) + C_1 \\
&= \operatorname{arctg} \left(\frac{x}{\sqrt{2 + x^2}} \right) + C_1 \\
&= -\operatorname{arctg} \left(\frac{\sqrt{x^2 + 2}}{x} \right) + C.
\end{aligned}$$

于是, 最后得到

$$\begin{aligned}
&\int \frac{\sqrt{x^2 + 2}}{x^2 + 1} dx = \ln(x + \sqrt{x^2 + 2}) \\
&\quad - \operatorname{arctg} \left(\frac{\sqrt{x^2 + 2}}{x} \right) + C.
\end{aligned}$$

化二次三项式为正则型, 以计算下列积分:

1961. $\int \frac{dx}{(x^2 + x + 1) \sqrt{x^2 + x - 1}}.$

解 $\int \frac{dx}{(x^2 + x + 1) \sqrt{x^2 + x - 1}}$

$$= \frac{1}{\sqrt{6}} \ln \left| \frac{(2x+1) \sqrt{2} + \sqrt{3(x^2+x-1)}}{(2x+1) \sqrt{2} - \sqrt{3(x^2+x-1)}} \right| + C.$$

1962. $\int \frac{x^2 dx}{(4-2x+x^2) \sqrt{2+2x-x^2}}.$

解 $\int \frac{x^2 dx}{(4-2x+x^2) \sqrt{2+2x-x^2}}$
 $= \int \frac{(x-1)^2 + 2(x-1) + 1}{(3+(x-1)^2) \sqrt{3-(x-1)^2}} dx.$

设 $x-1 = \sqrt{3} \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$dx = \sqrt{3} \cos t dt, \sqrt{2+2x-x^2} = \sqrt{3} \cos t.$$

代入得

$$\begin{aligned} & \int \frac{x^2 dx}{(4-2x+x^2) \sqrt{2+2x-x^2}} \\ &= \int \frac{1+2\sqrt{3}\sin t+3\sin^2 t}{3(1+\sin^2 t)} dt \\ &= \int dt + \frac{2}{\sqrt{3}} \int \frac{\sin t}{1+\sin^2 t} dt - \frac{2}{3} \int \frac{dt}{1+\sin^2 t} \\ &= t - \frac{2}{\sqrt{3}} \int \frac{d(\cos t)}{2-\cos^2 t} - \frac{2}{3} \int \frac{d(\operatorname{tg} t)}{1+2\operatorname{tg}^2 t} \\ &= t - \frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{2} + \cos t}{\sqrt{2} - \cos t} \right| \\ &\quad - \frac{\sqrt{2}}{3} \operatorname{arctg}(\sqrt{2} \operatorname{tg} t) + C \end{aligned}$$

$$= \arcsin \frac{x-1}{\sqrt{3}} - \frac{1}{\sqrt{6}} \ln \frac{\sqrt{6} + \sqrt{2+2x-x^2}}{\sqrt{6} - \sqrt{2+2x-x^2}}$$

$$- \frac{\sqrt{2}}{3} \operatorname{arc} \operatorname{tg} \frac{(x-1)\sqrt{2}}{\sqrt{2+2x-x^2}} + C.$$

1963. $\int \frac{(x+1)dx}{(x^2+x+1)\sqrt{x^2+x+1}}.$

解 $\int \frac{(x+1)dx}{(x^2+x+1)\sqrt{x^2+x+1}}$

$$= \int \frac{\left(x + \frac{1}{2}\right) + \frac{1}{2}}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}} d\left(x + \frac{1}{2}\right)$$

$$= \int \frac{\left(x + \frac{1}{2}\right) d\left(x + \frac{1}{2}\right)}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}} + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}}$$

$$= \frac{1}{2} \int \frac{d\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}} + \frac{1}{2} \cdot \frac{x + \frac{1}{2}}{\frac{3}{4} \sqrt{x^2+x+1}} \quad *)$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2x+1}{3\sqrt{x^2+x+1}} + C$$

$$= \frac{2(x-1)}{3\sqrt{x^2+x+1}} + C.$$

*) 利用 1781 题的结果.

1964.⁺ 利用线性分式的代换 $x = \frac{\alpha + \beta t}{1+t}$, 计算积分:

$$\int \frac{dx}{(x^2-x+1)\sqrt{x^2+x+1}}.$$

解 线性分式的代换

$$x = \frac{\alpha + \beta t}{1+t}$$

给出

$$x^2 \pm x + 1$$

$$= \frac{(\beta^2 \pm \beta + 1)t^2 + [2\alpha\beta \pm (\alpha + \beta) + 2]t + (\alpha^2 \pm \alpha + 1)}{(1+t)^2}$$

要求 $2\alpha\beta \pm (\alpha + \beta) + 2 = 0$ 即化成正则型. 当 $\alpha + \beta = 0$ 及 $\alpha\beta = -1$ 时即得上式. 例如, 取

$$\alpha = -1, \beta = 1,$$

我们有

$$x = \frac{t-1}{1+t} \text{ 或 } t = \frac{1+x}{1-x},$$

$$dx = \frac{2dt}{(1+t)^2}, x^2 - x + 1 = \frac{t^2 + 3}{(t+1)^2},$$

$$\sqrt{x^2 - x + 1} = \frac{\sqrt{1+3t^2}}{t+1},$$

其中不妨设 $t+1 > 0$.

于是,

$$\begin{aligned} & \int \frac{dx}{(x^2 - x + 1) \sqrt{x^2 - x + 1}} \\ &= 2 \int \frac{t+1}{(t^2+3) \sqrt{1+3t^2}} dt \\ &= 2 \int \frac{t dt}{(t^2+3) \sqrt{1+3t^2}} + 2 \int \frac{dt}{(t^2+3) \sqrt{1+3t^2}} \\ &= 2(I_1 + I_2). \end{aligned}$$

对于 I_1 , 设 $u = \sqrt{1+3t^2}$, 则

$$du = \frac{3t dt}{\sqrt{1+3t^2}}, t^2 + 3 = \frac{u^2 + 8}{3}.$$

代入得

$$\begin{aligned}
 I_1 &= \int \frac{t dt}{(t^2 + 3) \sqrt{1 + 3t^2}} = \int \frac{du}{u^2 + 8} \\
 &= \frac{1}{2\sqrt{2}} \arctg \left(\frac{u}{2\sqrt{2}} \right) + C_1 \\
 &= \frac{1}{2\sqrt{2}} \arctg \left[\frac{\sqrt{x^2 + x + 1}}{(1 - x)\sqrt{2}} \right] + C_1.
 \end{aligned}$$

对于 I_2 , 设 $u = \frac{3t}{\sqrt{1 + 3t^2}}$, 则

$$\frac{dt}{\sqrt{1 + 3t^2}} = \frac{du}{3 - u^2}, t^2 + 3 = \frac{27 - 8u^2}{3(3 - u^2)},$$

代入得

$$\begin{aligned}
 I_2 &= \int \frac{dt}{(t^2 + 3) \sqrt{1 + 3t^2}} = 3 \int \frac{du}{27 - 8u^2} \\
 &= \frac{1}{4\sqrt{6}} \ln \left| \frac{3\sqrt{3} + 2\sqrt{2}u}{3\sqrt{3} - 2\sqrt{2}u} \right| + C_2 \\
 &= \frac{1}{4\sqrt{6}} \ln \left| \frac{\sqrt{3(x^2 + x + 1)} + (x + 1)\sqrt{2}}{\sqrt{3(x^2 + x + 1)} - (x + 1)\sqrt{2}} \right| \\
 &\quad + C_2 \\
 &= \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{x^2 - x + 1}}{\sqrt{3(x^2 + x + 1)} - (x + 1)\sqrt{2}} \right| \\
 &\quad + C_2.
 \end{aligned}$$

于是, 最后得到

$$\begin{aligned}
 &\int \frac{dx}{(x^2 - x + 1) \sqrt{x^2 + x + 1}} \\
 &= -\frac{1}{\sqrt{2}} \operatorname{arctg} \left(\frac{\sqrt{x^2 + x + 1}}{(x - 1)\sqrt{2}} \right)
 \end{aligned}$$

$$+ \frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{x^2 - x + 1}}{\sqrt{3(x^2 + x + 1)} - (x + 1)\sqrt{2}} \right| + C.$$

1965⁺. 求

$$\int \frac{dx}{(x^2 + 2)\sqrt{2x^2 - 2x + 5}}.$$

解 此题与 1964 题均属于下述类型的积分

$$\int \frac{Mx + N}{(x^2 + px + q)^m \sqrt{ax^2 + bx + c}}$$

【参看微积分学教程(Г. М. 菲赫金哥尔茨)第二卷第一分册 55 页“272. 其它的计算方法”】

设 $x = \frac{\alpha + \beta t}{1 + t}$, 适当选择 α 与 β , 使得在两个三

项式中同时消去一次项. 为此, 将 $x = \frac{\alpha + \beta t}{1 + t}$ 分别代入 $x^2 + 2$ 及 $2x^2 - 2x + 5$ 中, 并令一次项的系数等于零, 求得

$$\alpha = -1, \beta = 2,$$

即设

$$x = \frac{2t - 1}{1 + t}.$$

从而有

$$dx = \frac{3}{(t + 1)^2} dt, x^2 + 2 = \frac{3(2t^2 + 1)}{(t + 1)^2},$$

$$\sqrt{2x^2 - 2x + 5} = \frac{3\sqrt{t^2 + 1}}{|t + 1|}.$$

以下不妨设 $t + 1 > 0$.

代入得

$$\begin{aligned}
& \int \frac{dx}{(x^2 + 2) \sqrt{2x^2 - 2x + 5}} \\
&= \frac{1}{3} \int \frac{t + 1}{(2t^2 + 1) \sqrt{t^2 + 1}} dt \\
&= \frac{1}{3} \int \frac{t dt}{(2t^2 + 1) \sqrt{t^2 + 1}} \\
&\quad + \frac{1}{3} \int \frac{dt}{(2t^2 + 1) \sqrt{t^2 + 1}}.
\end{aligned}$$

对于右端的第一个积分, 设 $u = \sqrt{t^2 + 1}$, 代入后

计算得

$$\begin{aligned}
& \frac{1}{3} \int \frac{t dt}{(2t^2 + 1) \sqrt{t^2 + 1}} = \frac{1}{3} \int \frac{du}{2u^2 - 1} \\
&= -\frac{1}{6\sqrt{2}} \ln \frac{\sqrt{2}u - 1}{\sqrt{2}u + 1} + C_1 \\
&= -\frac{1}{6\sqrt{2}} \ln \frac{\sqrt{2}(2x^2 - 2x + 5) + (x - 2)}{\sqrt{2}(2x^2 - 2x + 5) - (x - 2)} + C_1.
\end{aligned}$$

对于右端的第二个积分, 设 $u = \frac{t}{\sqrt{t^2 + 1}}$, 代入后
计算得

$$\begin{aligned}
& \frac{1}{3} \int \frac{dt}{(2t^2 + 1) \sqrt{t^2 + 1}} = \frac{1}{3} \int \frac{du}{1 + u^2} \\
&= \frac{1}{3} \operatorname{arctg} u + C_2 = \frac{1}{3} \operatorname{arctg} \left(\frac{1 + x}{\sqrt{2x^2 - 2x + 5}} \right) + C_2 \\
&= -\frac{1}{3} \operatorname{arctg} \left(\frac{\sqrt{2x^2 - 2x + 5}}{x + 1} \right) + C_3.
\end{aligned}$$

于是, 最后得到

$$\begin{aligned} & \int \frac{dx}{(x^2 + 2) \sqrt{2x^2 - 2x + 5}} \\ &= \frac{1}{6\sqrt{2}} \ln \frac{\sqrt{2(2x^2 - 2x + 5)} + (x - 2)}{\sqrt{2(2x^2 - 2x + 5)} - (x - 2)} \\ & \quad - \frac{1}{3} \operatorname{arctg} \left(\frac{\sqrt{2x^2 - 2x + 5}}{x + 1} \right) + C. \end{aligned}$$

利用尤拉代换

(1) 若 $a > 0$, $\sqrt{ax^2 + bx + c} = \pm \sqrt{ax} + z$;

(2) 若 $c > 0$, $\sqrt{ax^2 + bx + c} = xz \pm \sqrt{c}$;

(3) $\sqrt{a(x - x_1)(x - x_2)} = z(x - x_1)$.

以求下列积分:

1966. $\int \frac{dx}{x + \sqrt{x^2 + x + 1}}.$

解 设 $\sqrt{x^2 + x + 1} = z - x$, 则

$$x = \frac{z^2 - 1}{1 + 2z}, dx = \frac{2(z^2 + z + 1)}{(1 + 2z)^2} dz,$$

$$\sqrt{x^2 + x + 1} = \frac{z^2 + z + 1}{1 + 2z}.$$

代入得

$$\begin{aligned} \int \frac{dx}{x + \sqrt{x^2 + x + 1}} &= \frac{1}{2} \int \frac{z^2 + z + 1}{z \left(z + \frac{1}{2} \right)^2} dz \\ &= \frac{1}{2} \int \left[\frac{4}{z} - \frac{3}{z + \frac{1}{2}} - \frac{3}{2 \left(z + \frac{1}{2} \right)^2} \right] dz \\ &= \frac{1}{2} \ln \frac{z^4}{\left| z + \frac{1}{2} \right|^3} + \frac{3}{4 \left(z + \frac{1}{2} \right)} + C_1 \end{aligned}$$

$$= \frac{1}{2} \ln \frac{z^4}{|2z+1|^3} + \frac{3}{2(2z+1)} + C,$$

其中 $z = x + \sqrt{x^2 + x + 1}$.

1967. $\int \frac{dx}{1 + \sqrt{1 - 2x - x^2}}.$

解 设 $\sqrt{1 - 2x - x^2} = xz - 1$, 则

$$z = \frac{1 + \sqrt{1 - 2x - x^2}}{x}, x = \frac{2(z - 1)}{z^2 + 1},$$

$$dx = \frac{2(1 + 2z - z^2)}{(z^2 + 1)^2} dz,$$

$$\sqrt{1 - 2x - x^2} + 1 = \frac{2z(z - 1)}{z^2 + 1},$$

代入得

$$\begin{aligned} \int \frac{dx}{1 + \sqrt{1 - 2x - x^2}} &= \int \frac{1 + 2z - z^2}{z(z - 1)(z^2 + 1)} dz \\ &= \int \left(\frac{1}{z - 1} - \frac{1}{z} - \frac{2}{z^2 + 1} \right) dz \\ &= \ln \left| \frac{z - 1}{z} \right| - 2 \operatorname{arctg} z + C, \end{aligned}$$

其中 $z = \frac{1 + \sqrt{1 - 2x - x^2}}{x}$.

1968. $\int x \sqrt{x^2 - 2x + 2} dx.$

解 设 $\sqrt{x^2 - 2x + 2} = z - x$, 则

$$x = \frac{z^2 - 2}{2(z - 1)}, dx = \frac{z^2 - 2z + 2}{2(z - 1)^2} dz,$$

$$\sqrt{x^2 - 2x + 2} = \frac{z^2 - 2z + 2}{2(z - 1)}.$$

代入得

$$\begin{aligned}
& \int x \sqrt{x^2 - 2x + 2} dx \\
&= \frac{1}{8} \int \frac{(z^2 - 2)(z^2 - 2z + 2)^2}{(z - 1)^4} dz \\
&= \frac{1}{8} \int \frac{[(z - 1)^2 + 2(z - 1) - 1] \cdot [(z - 1)^2 + 1]^2}{(z - 1)^4} dz \\
&= \frac{1}{8} \int \left\{ \left[(z - 1)^2 - (z - 1)^{-4} \right] \right. \\
&\quad + \left[2(z - 1) + 2(z - 1)^{-3} \right] \\
&\quad + \left[1 - (z - 1)^{-2} \right] + 4(z - 1)^{-1} \left. \right\} d(z - 1) \\
&= \frac{1}{8} \left\{ \frac{1}{3} \left[(z - 1)^3 + (z - 1)^{-3} \right] \right. \\
&\quad + \left[(z - 1)^2 - (z - 1)^{-2} \right] \\
&\quad + \left. \left[(z - 1) + (z - 1)^{-1} \right] \right\} + \frac{1}{2} \ln |z - 1| + C,
\end{aligned}$$

其中 $z = x + \sqrt{x^2 - 2x + 2}$.

1969. $\int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx.$

解 设 $\sqrt{x^2 + 3x + 2} = z(x + 1)$, 则

$$x = \frac{2 - z^2}{z^2 - 1}, dx = -\frac{2z}{(z^2 - 1)^2} dz,$$

$$\sqrt{x^2 + 3x + 2} = \frac{z}{z^2 - 1}.$$

代入得

$$\begin{aligned}
& \int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx \\
&= \int \frac{2z(2 - z - z^2)}{(z^2 - z - 2)(z^2 - 1)^2} dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{5\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}}{2} + z + \frac{1}{2}}{\frac{\sqrt{5}}{2} - z - \frac{1}{2}} \right| \\
&\quad + \frac{2}{1-z-z^2} - \frac{4(2z+1)}{5(1-z-z^2)} + C \\
&= \frac{2}{5\sqrt{5}} \ln \left| \frac{\sqrt{5} + 2z + 1}{\sqrt{5} - 2z - 1} \right| + \frac{2(3-4z)}{5(1-z-z^2)} + C,
\end{aligned}$$

其中 $z = \sqrt{x(1+x)} - x$.

*) 利用 1921 题的递推公式.

利用各种方法, 计算下列积分:

1971. $\int \frac{dx}{\sqrt{x^2+1} - \sqrt{x^2-1}}.$

解
$$\begin{aligned}
&\int \frac{dx}{\sqrt{x^2+1} - \sqrt{x^2-1}} \\
&= \int \frac{\sqrt{x^2+1} + \sqrt{x^2-1}}{(x^2+1) - (x^2-1)} dx \\
&= \frac{1}{2} \int \sqrt{x^2+1} dx + \frac{1}{2} \int \sqrt{x^2-1} dx \\
&= \frac{x}{4} (\sqrt{x^2+1} + \sqrt{x^2-1}) \\
&\quad + \frac{1}{4} \ln \left| \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2-1}} \right| + C.
\end{aligned}$$

1972. $\int \frac{xdx}{(1-x^3)\sqrt{1-x^2}}.$

解 设 $\frac{1+x}{1-x} = z$, 则

$$x = \frac{z-1}{z+1}, dx = \frac{2}{(z+1)^2} dz,$$

代入得

$$\begin{aligned}
 & \int \frac{x dx}{(1-x^3) \sqrt{1-x^2}} \\
 &= \frac{1}{2} \int \frac{(z^2-1) dz}{\sqrt{z} (3z^2+1)} \\
 &= \int \frac{(z^2-1) d(\sqrt{z})}{3z^2+1} \\
 &= \int \left(\frac{1}{3} - \frac{4}{3(3z^2+1)} \right) d(\sqrt{z}) \\
 &= \frac{\sqrt{z}}{3} - \frac{4}{3} \cdot \frac{1}{\sqrt[4]{3}} \int \frac{d(\sqrt[4]{3z^2})}{(\sqrt[4]{3z^2})^4 + 1} \\
 &= \frac{\sqrt{z}}{3} - \frac{4}{3 \sqrt[4]{3}} \left[\frac{1}{4 \sqrt{2}} \ln \frac{z \sqrt{3} + \sqrt[4]{12z^2} + 1}{z \sqrt{3} - \sqrt[4]{12z^2} + 1} \right. \\
 &\quad \left. + \frac{1}{2 \sqrt{2}} \operatorname{arctg} \left(\frac{\sqrt[4]{12z^2}}{1 - z \sqrt{3}} \right) \right] + C \\
 &= \frac{\sqrt{z}}{3} - \frac{1}{3 \sqrt[4]{12}} \left[\ln \frac{z \sqrt{3} + \sqrt[4]{12z^2} + 1}{z \sqrt{3} - \sqrt[4]{12z^2} + 1} \right. \\
 &\quad \left. - 2 \operatorname{arctg} \left(\frac{\sqrt[4]{12z^2}}{z \sqrt{3} - 1} \right) \right] + C,
 \end{aligned}$$

其中 $z = \frac{1+x}{1-x}$.

*) 利用 1884 题的结果.

1973. $\int \frac{dx}{\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}.$

解 $\int \frac{dx}{\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}$
 $= \int \frac{-\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}{(\sqrt{2} + \sqrt{1-x} + \sqrt{1+x})}$

$$\begin{aligned}
& \cdot \frac{1}{(-\sqrt{2} + \sqrt{1-x} + \sqrt{1+x})} dx \\
&= \int \frac{-\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}{2\sqrt{1-x^2}} dx \\
&= -\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \int \frac{dx}{\sqrt{1+x}} \\
&\quad + \frac{1}{2} \int \frac{dx}{\sqrt{1-x}} \\
&= -\frac{1}{\sqrt{2}} \arcsin x + \sqrt{1+x} - \sqrt{1-x} + C.
\end{aligned}$$

1974. $\int \frac{x + \sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}} dx.$

解
$$\begin{aligned}
& \int \frac{x + \sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}} dx \\
&= \int \frac{(x + \sqrt{1+x+x^2})(1+x - \sqrt{1+x+x^2})}{(1+x)^2 - (1+x+x^2)} dx \\
&= \int \frac{\sqrt{1+x+x^2} - 1}{x} dx \\
&= \int \frac{\sqrt{1+x+x^2}}{x} dx - \ln|x|.
\end{aligned}$$

对于积分 $\int \frac{\sqrt{1+x+x^2}}{x} dx$, 设 $x = \frac{1}{t}$, 则

$$dx = -\frac{1}{t^2} dt, \quad \sqrt{1+x+x^2} = \frac{\sqrt{t^2+t+1}}{|t|}.$$

不妨设 $t > 0$, 代入得

$$\begin{aligned}
& \int \frac{\sqrt{1+x+x^2}}{x} dx \\
&= -\int \frac{\sqrt{t^2+t+1}}{t^2} dt
\end{aligned}$$

$$\begin{aligned}
&= \int \sqrt{t^2 + t + 1} d\left(\frac{1}{t}\right) \\
&= \frac{\sqrt{t^2 + t + 1}}{t} - \frac{1}{2} \int \frac{2t + 1}{t \sqrt{1 + t + t^2}} dt \\
&= \sqrt{x^2 + x + 1} - \int \frac{dt}{\sqrt{1 + t + t^2}} \\
&\quad - \frac{1}{2} \int \frac{dt}{t \sqrt{1 + t + t^2}} \\
&= \sqrt{x^2 + x + 1} - \ln\left(t + \frac{1}{2} + \sqrt{1 + t + t^2}\right) \\
&\quad + \frac{1}{2} \int \frac{d\left(\frac{1}{t}\right)}{\sqrt{\left(\frac{1}{t}\right)^2 + \left(\frac{1}{t}\right) + 1}} \\
&= \sqrt{x^2 + x + 1} - \ln \frac{2 + x + 2\sqrt{1 + x + x^2}}{2x} \\
&\quad + \frac{1}{2} \ln \left[\frac{1}{t} + \frac{1}{2} + \sqrt{\frac{1}{t^2} + \frac{1}{t} + 1} \right] + C_1 \\
&= \sqrt{x^2 + x + 1} - \ln \frac{2 + x + 2\sqrt{1 + x + x^2}}{2x} \\
&\quad + \frac{1}{2} \ln \frac{2x + 1 + 2\sqrt{1 + x + x^2}}{2} + C_1 \\
&= \sqrt{x^2 + x + 1} + \frac{1}{2} \ln \frac{2x + 1 + 2\sqrt{1 + x + x^2}}{(2 + x + 2\sqrt{1 + x + x^2})^2} \\
&\quad + \ln x + C.
\end{aligned}$$

于是,当 $x > 0$ 时,最后得到

$$\begin{aligned}
&\int \frac{x + \sqrt{1 + x + x^2}}{1 + x + \sqrt{1 + x + x^2}} dx \\
&= \sqrt{x^2 + x + 1}
\end{aligned}$$

$$+ \frac{1}{2} \ln \frac{2x+1+2\sqrt{1+x+x^2}}{(2+x+2\sqrt{1+x+x^2})^2} + C,$$

当 $x < 0$ 时, 可获同样的结果.

1975. $\int \frac{\sqrt{x(x+1)}}{\sqrt{x} + \sqrt{x+1}} dx.$

解
$$\begin{aligned} & \int \frac{\sqrt{x(x+1)}}{\sqrt{x} + \sqrt{x+1}} dx \\ &= \int \frac{\sqrt{x(x+1)} \cdot (\sqrt{x+1} - \sqrt{x})}{(x+1) - x} dx \\ &= \int [(x+1)\sqrt{x} - x\sqrt{x+1}] dx \\ &= \int [x^{\frac{3}{2}} + x^{\frac{1}{2}} - (x+1)^{\frac{3}{2}} + (x+1)^{\frac{1}{2}}] dx \\ &= \frac{2}{3} [(x+1)^{\frac{3}{2}} + x^{\frac{3}{2}}] - \frac{2}{5} [(x+1)^{\frac{5}{2}} - x^{\frac{5}{2}}] + C. \end{aligned}$$

1976. $\int \frac{(x^2-1)dx}{(x^2+1)\sqrt{x^4+1}}.$

解
$$\begin{aligned} \int \frac{(x^2-1)dx}{(x^2+1)\sqrt{x^4+1}} &= \int \frac{\frac{x^2-1}{(x^2+1)^2} dx}{\sqrt{\frac{x^4+1}{(x^2+1)^2}}} \\ &= \int \frac{\frac{x^2-1}{(x^2+1)^2} dx}{\sqrt{1 - \left(\frac{x\sqrt{2}}{x^2+1}\right)^2}} \end{aligned}$$

下面我们先考虑积分 $\int \frac{x^2-1}{(x^2+1)^2} dx$, 设 $x = \operatorname{tgt}$,

$-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则有 $dx = \sec^2 t dt$.

代入得

$$\begin{aligned}\int \frac{x^2 - 1}{(x^2 + 1)^2} dx &= \int \frac{\operatorname{tg}^2 t - 1}{\sec^4 t} \cdot \sec^2 t dt \\&= \int (\sin^2 t - \cos^2 t) dt = - \int \cos 2t dt \\&= -\frac{1}{2} \sin 2t + C_1 = -\frac{x}{1+x^2} + C_1.\end{aligned}$$

从而, 可得 $\frac{x^2 - 1}{(x^2 + 1)^2} dx = -\frac{1}{\sqrt{2}} d\left(\frac{x\sqrt{2}}{1+x^2}\right)$.

于是,

$$\begin{aligned}&\int \frac{(x^2 - 1) dx}{(x^2 + 1) \sqrt{x^4 + 1}} \\&= -\frac{1}{\sqrt{2}} \int \frac{d\left(\frac{x\sqrt{2}}{1+x^2}\right)}{\sqrt{1 - \left(\frac{x\sqrt{2}}{1+x^2}\right)^2}} \\&= -\frac{1}{\sqrt{2}} \arcsin\left(\frac{x\sqrt{2}}{1+x^2}\right) + C.\end{aligned}$$

1977. $\int \frac{x^2 + 1}{(x^2 - 1) \sqrt{x^4 + 1}} dx.$

解 仿照 1976 题, 可得

$$\begin{aligned}\int \frac{x^2 + 1}{(x^2 - 1) \sqrt{x^4 + 1}} dx &= \int \frac{\frac{x^2 + 1}{(x^2 - 1)^2}}{\sqrt{\frac{x^4 + 1}{(x^2 - 1)^2}}} dx \\&= -\frac{1}{\sqrt{2}} \int \frac{d\left(\frac{x\sqrt{2}}{x^2 - 1}\right)}{\sqrt{1 + \left(\frac{x\sqrt{2}}{x^2 - 1}\right)^2}}\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\sqrt{2}} \ln \left| \frac{x\sqrt{2}}{x^2-1} + \sqrt{1 + \left(\frac{x\sqrt{2}}{x^2-1} \right)^2} \right| + C \\
&= -\frac{1}{\sqrt{2}} \ln \left| \frac{x\sqrt{2} + \sqrt{x^4+1}}{x^2-1} \right| + C.
\end{aligned}$$

1978. $\int \frac{dx}{x \sqrt{x^4 + 2x^2 - 1}}.$

解 作变换 $\frac{1}{x} = \sqrt{t}$ (这里设 $x > 0$. 若 $x < 0$, 则作变换 $\frac{1}{x} = -\sqrt{t}$. 最后结果相同), 则

$$dx = -\frac{1}{2t\sqrt{t}} dt, \quad \sqrt{x^4 + 2x^2 - 1} = \frac{\sqrt{1 + 2t - t^2}}{t}.$$

代入得

$$\begin{aligned}
\int \frac{dx}{x \sqrt{x^4 + 2x^2 - 1}} &= -\frac{1}{2} \int \frac{dt}{\sqrt{1 + 2t - t^2}} \\
&= \frac{1}{2} \int \frac{d(1-t)}{\sqrt{2 - (1-t)^2}} \\
&= \frac{1}{2} \arcsin \left(\frac{1-t}{\sqrt{2}} \right) + C \\
&= \frac{1}{2} \arcsin \left(\frac{x^2 - 1}{x^2 \sqrt{2}} \right) + C \quad (|x| > \sqrt{\sqrt{2} - 1}).
\end{aligned}$$

1979. $\int \frac{(x^2 + 1)dx}{x \sqrt{x^4 + x^2 + 1}}.$

解
$$\begin{aligned}
&\int \frac{(x^2 + 1)dx}{x \sqrt{x^4 + x^2 + 1}} \\
&= \int \frac{xdx}{\sqrt{x^4 + x^2 + 1}} + \int \frac{dx}{x \sqrt{x^4 + x^2 + 1}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{d\left(x^2 + \frac{1}{2}\right)}{\sqrt{\left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\
&\quad - \frac{1}{2} \int \frac{d\left(\frac{1}{x^2}\right)}{\sqrt{\left(\frac{1}{x^2} + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\
&= \frac{1}{2} \ln \frac{x^2 + \frac{1}{2} + \sqrt{x^4 + x^2 + 1}}{\frac{1}{x^2} + \frac{1}{2} + \sqrt{\frac{x^4 + x^2 + 1}{x^4}}} + C \\
&= \frac{1}{2} \ln \frac{x^2(1 + 2x^2 + 2\sqrt{x^4 + x^2 + 1})}{2 + x^2 + 2\sqrt{x^4 + x^2 + 1}} + C.
\end{aligned}$$

1980. 证明积分

$$\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$$

(式中 R 为有理函数) 的求法, 归结为有理函数的积分法.

证 当 a, c 中至少有一个为零时, 则积分

$$\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$$

的求法显然可归结为有理函数的积分法.

当 $a \neq 0, c \neq 0$ 时, 设 $\sqrt{ax+b} = z$, 则

$$x = \frac{z^2 - b}{a}, dx = \frac{2}{a} z dz,$$

$$\sqrt{cx+d} = \sqrt{\frac{c}{a} z^2 + d - \frac{bc}{a}} = \sqrt{c_1 z^2 + d_1},$$

式中 $c_1 = \frac{c}{a}, d_1 = d - \frac{bc}{a}$.

代入得

$$\begin{aligned} & \int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx \\ &= \int R\left(\frac{z^2-b}{a}, z, \sqrt{c_1 z^2 + d_1}\right) \frac{2}{a} z dz, \\ &= \int R_1(z, \sqrt{c_1 z^2 + d_1}) dz, \end{aligned}$$

其中 R_1 为有理函数.

再设 $\sqrt{c_1 z^2 + d_1} = \pm \sqrt{c_1} z + u (c_1 > 0)$ 或 $\sqrt{c_1 z^2 + d_1} = zu \pm \sqrt{d_1} (d_1 > 0)$ —— 尤拉代换, 就可将被积函数有理化. 于是, 积分

$$\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$$

的求法可归结为有理函数的积分法.

二项微分式

$$\int x^m (a + bx^n)^p dx,$$

(式中 m, n 和 p 为有理数) 仅在下列三种情形可化为有理函数的积分(契比协夫定理):

第一种情形, p 为整数. 假定 $x = z^N$, 其中 N 为分数 m 和 n 的公分母.

第二种情形, $\frac{m+1}{n}$ 为整数. 假定 $a + bx^n = z^N$, 其中 N 为分数 p 的分母.

第三种情形, $\frac{m+1}{n} + p$ 为整数. 利用代换: ax^{-n}

$$+ \frac{1}{8} \ln(\sqrt{x} + \sqrt{1+x}) + C \quad (x > 0).$$

*) 利用 1921 题的结果.

$$1982. \int \frac{\sqrt{x}}{(1 + \sqrt[3]{x})^2} dx.$$

$$\text{解} \quad \frac{\sqrt{x}}{(1 + \sqrt[3]{x})^2} = x^{\frac{1}{2}}(1 + x^{\frac{1}{3}})^{-2}, m = \frac{1}{2}, n = \frac{1}{3},$$

$p = -2$; p 为整数, 这是二项微分式的第一种情形.

设 $x = z^6$, 则

$$dx = 6z^5 dz, \sqrt{x} = z^3, \sqrt[3]{x} = z^2.$$

代入得

$$\begin{aligned} \int \frac{\sqrt{x}}{(1 + \sqrt[3]{x})^2} dx &= 6 \int \frac{z^3}{(z^2 + 1)^2} dz \\ &= 6 \int \left[z^4 - 2z^2 + 3 - \frac{4}{z^2 + 1} + \frac{1}{(z^2 + 1)^2} \right] dz \\ &= \frac{6}{5} z^5 - 4z^3 + 18z - 24 \operatorname{arc} \operatorname{tg} z \\ &\quad + 6 \left(\frac{z}{2(z^2 + 1)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} z \right)^{**} + C \\ &= \frac{6}{5} x^{\frac{5}{6}} - 4x^{\frac{1}{2}} + 18x^{\frac{1}{6}} + \frac{3x^{\frac{1}{6}}}{1 + x^{\frac{1}{3}}} - 21 \operatorname{arc} \operatorname{tg}(x^{\frac{1}{6}}) \\ &\quad + C. \end{aligned}$$

*) 利用 1921 题的结果.

$$1983. \int \frac{x dx}{\sqrt{1 + \sqrt[3]{x^2}}}.$$

$$\text{解} \quad \frac{x}{\sqrt{1 + \sqrt[3]{x^2}}} = x(1 + x^{\frac{2}{3}})^{-\frac{1}{2}}, m = 1, n = \frac{2}{3}, p$$

$= -\frac{1}{2}; \frac{m+1}{n} = 3$, 这是二项微分式的第二种情形.

设 $1 + x^{\frac{2}{3}} = z^2$, 则

$$x = (z^2 - 1)^{\frac{3}{2}}, dx = 3z(z^2 - 1)^{\frac{1}{2}} dz.$$

代入得

$$\begin{aligned} \int \frac{x dx}{\sqrt{1 + \sqrt[3]{x^2}}} &= 3 \int (z^2 - 1)^2 dz \\ &= \frac{3}{5} z^5 - 2z^3 + 3z + C, \end{aligned}$$

其中 $z = \sqrt{1 + \sqrt[3]{x}}$.

1984. $\int \frac{x^5 dx}{\sqrt{1-x^2}}$

解 $\frac{x^5}{\sqrt{1-x^2}} = x^5(1-x^2)^{-\frac{1}{2}}, m=5, n=2, p=$
 $-\frac{1}{2}; \frac{m+1}{n} = 3$, 这是二项微分式的第二种情形.

设 $\sqrt{1-x^2} = z$ (不妨设 $x > 0$), 则

$$x = \sqrt{1-z^2}, dx = -\frac{z}{\sqrt{1-z^2}} dz.$$

代入得

$$\begin{aligned} \int \frac{x^5 dx}{\sqrt{1-x^2}} &= - \int (1-z^2)^2 dz \\ &= -z + \frac{2}{3} z^3 - \frac{1}{5} z^5 + C, \end{aligned}$$

其中 $z = \sqrt{1-x^2}$.

1985. $\int \frac{dx}{\sqrt[3]{1+x^3}}.$

解 $\frac{1}{\sqrt[3]{1+x^3}} = x^0(1+x^3)^{-\frac{1}{3}}, m=0, n=3,$

$p = -\frac{1}{3}; \frac{m+1}{n} + p = 0$, 这是二项微分式的第三种情形.

设 $x^{-3} + 1 = z^3$, 则

$$x = (z^3 - 1)^{-\frac{1}{3}}, dx = -z^2(z^3 - 1)^{-\frac{4}{3}}dz.$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[3]{1+x^3}} &= - \int \frac{z}{z^3-1} dz \\ &= -\frac{1}{3} \int \frac{dz}{z-1} + \frac{1}{3} \int \frac{z-1}{z^2+z+1} dz \\ &= -\frac{1}{3} \ln|z-1| + \frac{1}{6} \ln(z^2+z+1) \\ &\quad - \frac{1}{\sqrt{3}} \operatorname{arctg}\left(\frac{2z+1}{\sqrt{3}}\right) + C \\ &= \frac{1}{6} \ln \frac{z^2+z+1}{(z-1)^2} - \frac{1}{\sqrt{3}} \operatorname{arctg}\left(\frac{2z+1}{\sqrt{3}}\right) + C \end{aligned}$$

其中 $z = \frac{\sqrt[3]{1+x^3}}{x}.$

1986. $\int \frac{dx}{\sqrt[4]{1+x^4}}.$

解 $\frac{1}{\sqrt[4]{1+x^4}} = x^0(1+x^4)^{-\frac{1}{4}}, m=0, n=4, p=$

$-\frac{1}{4}; \frac{m+1}{n} + p = 0$, 这是二项微分式的第三种情

形.

设 $x^{-4} + 1 = z^4$, 则

$$z = \frac{\sqrt[4]{1+x^4}}{x} (z > 0, x > 0),$$

$$x = (z^4 - 1)^{-\frac{1}{4}}, dx = -z^3(z^4 - 1)^{-\frac{5}{4}} dz.$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[4]{1+x^4}} &= - \int \frac{z^3}{z^4 - 1} dz \\ &= \int \left(\frac{1}{4(z+1)} - \frac{1}{4(z-1)} - \frac{1}{2(z^2+1)} \right) dz \\ &= \frac{1}{4} \ln \left| \frac{z+1}{z-1} \right| - \frac{1}{2} \arctan z + C, \end{aligned}$$

其中 $z = \frac{\sqrt[4]{1+x^4}}{x}$.

1987⁺ . $\int \frac{dx}{x \sqrt[6]{1+x^6}}.$

解 $\frac{dx}{x \sqrt[6]{1+x^6}} = x^{-1}(1+x^6)^{-\frac{1}{6}}, m = -1, n = 6,$

$p = -\frac{1}{6}; \frac{m+1}{n} = 0$, 这是二项微分式的第二种情形.

设 $1+x^6 = z^6$, 则

$$z = \sqrt[6]{1+x^6} (z > 0, x > 0),$$

$$x = \sqrt[6]{z^6-1}, dx = z^5(z^6-1)^{-\frac{5}{6}} dz.$$

代入得

$$\begin{aligned} \int \frac{dx}{x \sqrt[6]{1+x^6}} &= \int \frac{z^4 dz}{z^6 - 1} \\ &= \int \left(-\frac{1}{6(z+1)} + \frac{z+1}{6(z^2-z+1)} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6(z-1)} + \frac{-z+1}{6(z^2+z+1)} \Big) dz \\
& = \frac{1}{6} \ln \frac{z-1}{z+1} + \frac{1}{12} \ln \frac{z^2-z+1}{z^2+z+1} \\
& \quad + \frac{1}{2\sqrt{3}} \left(\operatorname{arctg} \left(\frac{2z-1}{\sqrt{3}} \right) + \operatorname{arctg} \left(\frac{2z+1}{\sqrt{3}} \right) \right) \\
& \quad + C_1 \\
& = \frac{1}{6} \ln \frac{z-1}{z+1} + \frac{1}{12} \ln \frac{z^2-z+1}{z^2+z+1} \\
& \quad + \frac{1}{2\sqrt{3}} \operatorname{arctg} \left(\frac{z^2-1}{z\sqrt{3}} \right) + C,
\end{aligned}$$

其中 $z = \sqrt[6]{1+x^6}$.

1988. $\int \frac{dx}{x^3 \sqrt[5]{1+\frac{1}{x}}}.$

解 $\frac{1}{x^3 \sqrt[5]{1+\frac{1}{x}}} = x^{-3}(1+x^{-1})^{-\frac{1}{5}}, m = -3,$

$n = -1, p = -\frac{1}{5}, \frac{m+1}{n} = 2$, 这是二项微分式的第二种情形.

设 $1+x^{-1} = z^5$, 则

$$x = (z^5 - 1)^{-1}, dx = -5z^4(z^5 - 1)^{-2}dz.$$

代入得

$$\begin{aligned}
\int \frac{dx}{x^3 \sqrt[5]{1+\frac{1}{x}}} &= -5 \int z^3(z^5 - 1)dz \\
&= -\frac{5}{9}z^9 + \frac{5}{4}z^4 + C,
\end{aligned}$$

其中 $z = \sqrt[5]{1 + \frac{1}{x}}$.

1989. $\int \sqrt[3]{3x - x^3} dx.$

解 $\sqrt[3]{3x - x^3} = x^{\frac{1}{3}}(3 - x^2)^{\frac{1}{3}}, m = \frac{1}{3}, n = 2,$

$p = \frac{1}{3}; \frac{m+1}{n} + p = 1$, 这是二项微分式的第三种情形.

设 $3x^{-2} - 1 = z^3$ (不妨设 $x > 0$), 则

$$z = \frac{\sqrt[3]{3x - x^3}}{x}, x = \frac{\sqrt{3}}{\sqrt{z^3 + 1}},$$

$$dx = -\frac{3\sqrt{3}}{2} \cdot \frac{z^2}{(z^3 + 1)^{\frac{3}{2}}} dz.$$

代入得

$$\begin{aligned} \int \sqrt[3]{3x - x^3} dx &= -\frac{9}{2} \int \frac{z^3}{(z^3 + 1)^2} dz \\ &= -\frac{9}{2} \int \frac{dz}{z^3 + 1} + \frac{9}{2} \int \frac{dz}{(z^3 + 1)^2} \\ &= -\frac{9}{2} \left[\frac{1}{6} \ln \frac{(z+1)^2}{z^2 - z + 1} + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2z-1}{\sqrt{3}} \right) \right]^{**} \\ &\quad + \frac{9}{2} \left[\frac{z}{3(z^3 + 1)} + \frac{1}{9} \ln \frac{(z+1)^2}{z^2 - z + 1} \right. \\ &\quad \left. + \frac{2}{3\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2z-1}{\sqrt{3}} \right) \right]^{***} + C \\ &= \frac{3z}{2(z^3 + 1)} - \frac{1}{4} \ln \frac{(z+1)^2}{z^2 - z + 1} \\ &\quad - \frac{\sqrt{3}}{2} \operatorname{arc} \operatorname{tg} \left(\frac{2z-1}{\sqrt{3}} \right) + C, \end{aligned}$$

其中 $z = \frac{\sqrt[3]{3x - x^3}}{x}$.

*) 利用 1881 题的结果.

* *) 利用 1892 题的结果.

1990. 在甚么情形下, 积分

$$\int \sqrt{1 + x^m} dx$$

(式中 m 为有理数) 为初等函数?

解 $\sqrt{1 + x^m} = x^0(1 + x^m)^{\frac{1}{2}}$, 由于 $p = \frac{1}{2}$, 故由契比协夫定理知, 仅在下述两种情形, 此函数的积分可化为有理函数的积分.

第一种情形, $\frac{1}{m}$ 为整数, 即 $m = \frac{1}{k_1} = \frac{2}{2k_1}$, 其中 $k_1 = \pm 1, \pm 2, \dots$;

第二种情形, $\frac{1}{m} + \frac{1}{2}$ 为整数, 即 $m = \frac{2}{2k_2 - 1}$, 其中 $k_2 = 0, \pm 1, \pm 2, \dots$.

综上所述, 即得: 当

$$m = \frac{2}{k}$$

(式中 $k = \pm 1, \pm 2, \dots$) 时, 积分

$$\int \sqrt{1 + x^m} dx$$

为初等函数.

§ 4. 三角函数的积分法

形如

$$\int \sin^m x \cos^n x dx$$

的积分(式中 m 及 n 为整数), 可利用巧妙的变换或运用递推公式计算.

求下列积分:

1991. $\int \cos^5 x dx.$

$$\begin{aligned}\text{解} \quad \int \cos^5 x dx &= \int \cos^4 x \cos x dx \\ &= \int (1 - \sin^2 x)^2 d(\sin x) \\ &= \int (1 - 2\sin^2 x + \sin^4 x) d(\sin x) \\ &= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C.\end{aligned}$$

1992. $\int \sin^6 x dx.$

$$\begin{aligned}\text{解} \quad \int \sin^6 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^3 dx \\ &= \frac{1}{8} \int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x) dx \\ &= \frac{x}{8} - \frac{3}{16} \sin 2x + \frac{3}{8} \int \frac{1 + \cos 4x}{2} dx \\ &\quad - \frac{1}{8} \int (1 - \sin^2 2x) \cos 2x dx \\ &= \frac{x}{8} - \frac{3}{16} \sin 2x + \frac{3x}{16} + \frac{3}{64} \sin 4x\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{16}\int(1-\sin^2 2x)d(\sin 2x) \\
& =\frac{5x}{16}-\frac{3}{16}\sin 2x+\frac{3}{64}\sin 4x \\
& \quad -\frac{1}{16}\sin 2x+\frac{1}{48}\sin^3 2x+C \\
& =\frac{5x}{16}-\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C
\end{aligned}$$

1993. $\int \cos^6 x dx$.

解
$$\begin{aligned}
\int \cos^6 x dx &= \int \sin^6\left(x - \frac{\pi}{2}\right) d\left(x - \frac{\pi}{2}\right) \\
&= \frac{5}{16}\left(x - \frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(x - \frac{\pi}{2}\right) \\
&\quad + \frac{3}{64}\sin 4\left(x - \frac{\pi}{2}\right) + \frac{1}{48}\sin^3 2\left(x - \frac{\pi}{2}\right) + C_1 \\
&= \frac{5x}{16} + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

*) 利用 1992 题的结果.

1994. $\int \sin^2 x \cos^4 x dx$.

解
$$\begin{aligned}
\int \sin^2 x \cos^4 x dx &= \frac{1}{4} \int \sin^2 2x \cos^2 x dx \\
&= \frac{1}{8} \int \sin^2 2x (1 + \cos 2x) dx \\
&= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx + \frac{1}{16} \int \sin^2 2x d(\sin 2x) \\
&= \frac{x}{16} - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C.
\end{aligned}$$

1995. $\int \sin^4 x \cos^5 x dx$.

解
$$\int \sin^4 x \cos^5 x dx = \int \sin^4 x (1 - \sin^2 x)^2 d(\sin x)$$

$$\begin{aligned}
&= -\frac{\operatorname{ctg} x}{\sin x} - \int \operatorname{ctg} x \frac{\cos x}{\sin^2 x} dx \\
&= -\frac{\cos x}{\sin^2 x} - \int \frac{1 - \sin^2 x}{\sin^3 x} dx \\
&= -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|,
\end{aligned}$$

于是,

$$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.$$

2000. $\int \frac{dx}{\cos^3 x}.$

$$\begin{aligned}
\text{解} \quad \int \frac{dx}{\cos^3 x} &= \int \frac{d\left(x + \frac{\pi}{2}\right)}{\sin^3\left(x + \frac{\pi}{2}\right)} \\
&= -\frac{\cos\left(x + \frac{\pi}{2}\right)}{2\sin^2\left(x + \frac{\pi}{2}\right)} + \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{x + \frac{\pi}{2}}{2} \right) \right|^{*}) + C \\
&= \frac{\sin x}{2\cos^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C.
\end{aligned}$$

*) 利用 1999 题的结果.

2001. $\int \frac{dx}{\sin^4 x \cos^4 x}.$

$$\begin{aligned}
\text{解} \quad \int \frac{dx}{\sin^4 x \cos^4 x} &= 16 \int \frac{dx}{\sin^4 2x} \\
&= -8 \int \csc^2 2x d(\operatorname{ctg} 2x) \\
&= -8 \int (1 + \operatorname{ctg}^2 2x) d(\operatorname{ctg} 2x) \\
&= -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x + C.
\end{aligned}$$

2002. $\int \frac{dx}{\sin^3 x \cos^5 x}.$

解 $\int \frac{dx}{\sin^3 x \cos^5 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos^5 x} dx$
 $= \int \frac{dx}{\sin x \cos^5 x} + \int \frac{dx}{\sin^3 x \cos^3 x}.$
 $= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^5 x} dx + \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos^3 x} dx$
 $= \int \frac{\sin x}{\cos^5 x} dx + 2 \int \frac{dx}{\sin x \cos^3 x} + \int \frac{dx}{\sin^3 x \cos x}$
 $= - \int \frac{d(\cos x)}{\cos^5 x} + 2 \int \frac{\sin x}{\cos^3 x} dx$
 $+ 3 \int \frac{dx}{\sin x \cos x} + \int \frac{\cos x}{\sin^3 x} dx$
 $= \frac{1}{4 \cos^4 x} - 2 \int \frac{d(\cos x)}{\cos^3 x} + 3 \int \frac{d(\operatorname{tg} x)}{\operatorname{tg} x} + \int \frac{d(\sin x)}{\sin^3 x}$
 $= \frac{1}{4 \cos^4 x} + \frac{1}{\cos^2 x} + 3 \ln |\operatorname{tg} x| - \frac{1}{2 \sin^2 x} + C_1$
 $= \frac{1}{4} \operatorname{tg}^4 x + \frac{3}{2} \operatorname{tg}^2 x - \frac{1}{2} \operatorname{ctg}^2 x + 3 \ln |\operatorname{tg} x| + C.$

2003. $\int \frac{dx}{\sin x \cos^4 x}.$

解 $\int \frac{dx}{\sin x \cos^4 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^4 x} dx$
 $= \int \frac{\sin x}{\cos^4 x} dx + \int \frac{dx}{\sin x \cos^2 x}$
 $= - \int \frac{d(\cos x)}{\cos^4 x} + \int \frac{\sin x}{\cos^2 x} dx + \int \frac{dx}{\sin x}$
 $= \frac{1}{3 \cos^3 x} - \int \frac{d(\cos x)}{\cos^2 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|$
 $= \frac{1}{3 \cos^3 x} + \frac{1}{\cos x} + \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.$

2004. $\int \operatorname{tg}^5 x dx.$

$$\begin{aligned}
 \text{解} \quad \int \operatorname{tg}^5 x dx &= \int \operatorname{tg} x (\sec^2 x - 1)^2 dx \\
 &= \int \sec^4 x \operatorname{tg} x dx - 2 \int \sec^2 x \operatorname{tg} x dx + \int \operatorname{tg} x dx \\
 &= \int \sec^3 x d(\sec x) - 2 \int \sec x d(\sec x) - \int \frac{d(\cos x)}{\cos x} \\
 &= \frac{1}{4} \sec^4 x - \sec^2 x - \ln |\cos x| + C_1 \\
 &= \frac{1}{4} \operatorname{tg}^4 x - \frac{1}{2} \operatorname{tg}^2 x - \ln |\cos x| + C.
 \end{aligned}$$

2005. $\int \operatorname{ctg}^6 x dx.$

$$\begin{aligned}
 \text{解} \quad \int \operatorname{ctg}^6 x dx &= \int \operatorname{ctg}^2 x (\csc^2 x - 1)^2 dx \\
 &= \int \operatorname{ctg}^2 x \csc^4 x dx - 2 \int \operatorname{ctg}^2 x \csc^2 x dx + \int \operatorname{ctg}^2 x dx \\
 &= - \int \operatorname{ctg}^2 x (1 + \operatorname{ctg}^2 x) d(\operatorname{ctg} x) \\
 &\quad + 2 \int \operatorname{ctg}^2 x d(\operatorname{ctg} x) + \int (\csc^2 x - 1) dx \\
 &= - \frac{1}{3} \operatorname{ctg}^3 x - \frac{1}{5} \operatorname{ctg}^5 x + \frac{2}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x - x + C \\
 &= - \frac{1}{5} \operatorname{ctg}^5 x + \frac{1}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x - x + C.
 \end{aligned}$$

2006. $\int \frac{\sin^4 x}{\cos^6 x} dx.$

$$\text{解} \quad \int \frac{\sin^4 x}{\cos^6 x} dx = \int \operatorname{tg}^4 x d(\operatorname{tg} x) = \frac{1}{5} \operatorname{tg}^5 x + C.$$

2007. $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}.$

$$\begin{aligned}
 \text{解} \quad & \int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = \int \frac{\sin^2 x dx}{\sqrt{\sin^3 x \cos^5 x}} \\
 & + \int \frac{\cos^2 x dx}{\sqrt{\sin^3 x \cos^5 x}} \\
 & = \int \sqrt{\operatorname{tg} x} d(\operatorname{tg} x) - \int \frac{d(\operatorname{ctg} x)}{\sqrt{\operatorname{ctg} x}} \\
 & = \frac{2}{3} \sqrt{\operatorname{tg}^3 x} - 2 \sqrt{\operatorname{ctg} x} + C.
 \end{aligned}$$

$$2008^+. \int \frac{dx}{\cos x \sqrt[3]{\sin^2 x}}.$$

解 设 $t = \sqrt[3]{\sin x}$, 不妨只考虑 $\cos x$ 为正的情况, 即

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \text{ 且 } x \neq 0, \text{ 则有}$$

$$dx = \frac{3t^2}{\sqrt{1-t^6}} dt, \cos x = \sqrt{1-t^6}.$$

代入得

$$\begin{aligned}
 & \int \frac{dx}{\cos x \sqrt[3]{\sin^2 x}} = 3 \int \frac{dt}{1-t^6} \\
 & = \frac{3}{2} \int \left(\frac{1}{1-t^3} + \frac{1}{1+t^3} \right) dt \\
 & = \frac{1}{2} \int \left(\frac{1}{1-t} + \frac{t+2}{1+t+t^2} \right) dt + \frac{3}{2} \int \frac{dt}{1+t^3} \\
 & = \frac{-1}{2} \ln |1-t| + \frac{1}{4} \int \frac{2t+1}{t^2+t+1} \\
 & \quad + \frac{3}{4} \int \frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{3}{2} \left[\frac{1}{6} \ln \frac{(t+1)^2}{t^2-t+1} \right. \\
 & \quad \left. + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2t-1}{\sqrt{3}} \right] + C
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \ln \frac{(t+1)^2(t^2+t+1)}{(1-t)^2(t^2-t+1)} \\
&\quad + \frac{\sqrt{3}}{2} \left[\operatorname{arctg} \left(\frac{2t+1}{\sqrt{3}} \right) + \operatorname{arctg} \left(\frac{2t-1}{\sqrt{3}} \right) \right] + C \\
&= \frac{1}{4} \ln \frac{(1+t)^3(1-t^3)}{(1-t)^3(1+t^3)} + \frac{\sqrt{3}}{2} \operatorname{arctg} \left(\frac{t\sqrt{3}}{1-t^2} \right) \\
&\quad + C,
\end{aligned}$$

其中 $t = \sqrt[3]{\sin x}$.

*) 利用 1881 题的结果.

2009. $\int \frac{dx}{\sqrt{\operatorname{tg} x}}.$

解 设 $t = \sqrt{\operatorname{tg} x}$, 则

$$x = \operatorname{arctg} t^2, dx = \frac{2t}{1+t^4} dt,$$

代入得

$$\begin{aligned}
\int \frac{dx}{\sqrt{\operatorname{tg} x}} &= 2 \int \frac{dt}{1+t^4} \\
&= 2 \left[\frac{1}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \right. \\
&\quad \left. + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t\sqrt{2}}{1-t^2} \right]^{*}) + C \\
&= \frac{1}{2\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} + \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{t\sqrt{2}}{1-t^2} + C.
\end{aligned}$$

其中 $t = \sqrt{\operatorname{tg} x}$.

*) 利用 1884 题的结果.

2010. $\int \frac{dx}{\sqrt[3]{\operatorname{tg} x}}.$

解 设 $\sqrt[3]{\operatorname{tg} x} = t$, 则

$$x = \operatorname{arctg} t^3, dx = \frac{3t^2}{1+t^6} dt,$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[3]{\operatorname{tg} x}} &= 3 \int \frac{tdt}{1+t^6} = \frac{3}{2} \int \frac{d(t^2)}{1+(t^2)^3} \\ &= \frac{3}{2} \left[\frac{1}{6} \ln \frac{(t^2+1)^2}{t^4-t^2+1} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2t^2-1}{\sqrt{3}} \right] + C \\ &= \frac{1}{4} \ln \frac{(t^2+1)^2}{t^4-t^2+1} + \frac{\sqrt{3}}{2} \operatorname{arctg} \frac{2t^2-1}{\sqrt{3}} + C, \end{aligned}$$

其中 $t = \sqrt[3]{\operatorname{tg} x}$.

*) 利用 1881 题的结果.

2011. 推出下列积分的递推公式

$$(a) I_n = \int \sin^n x dx; \quad (b) K_n = \int \cos^n x dx \quad (n > 2).$$

并利用推得的公式来计算

$$\int \sin^6 x dx \text{ 及 } \int \cos^8 x dx.$$

$$\begin{aligned} \text{解 } (a) I_n &= \int \sin^n x dx = - \int \sin^{n-1} x d(\cos x) \\ &= - \cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x dx \\ &= - \cos x \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\ &= - \cos x \sin^{n-1} x + (n-1) I_{n-2} + (1-n) I_n, \end{aligned}$$

于是,

$$I_n = - \frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2};$$

利用此公式及

$$I_0 = \int dx = x + C,$$

即得

$$\begin{aligned} I_6 &= \int \sin^6 x dx = -\frac{\cos x \sin^5 x}{6} + \frac{5}{6} I_4 \\ &= -\frac{\cos x \sin^5 x}{6} - \frac{5 \cos x \sin^3 x}{24} + \frac{5}{8} I_2 \\ &= -\frac{\cos x \sin^5 x}{6} - \frac{5 \cos x \sin^3 x}{24} \\ &\quad - \frac{5 \cos x \sin x}{16} + \frac{5}{16} x + C. \end{aligned}$$

$$\begin{aligned} (6) \quad K_n &= \int \cos^n x dx = \int \cos^{n-1} x d(\sin x) \\ &= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x dx \\ &= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx \\ &= \sin x \cos^{n-1} x + (n-1) K_{n-2} - (n-1) K_n \end{aligned}$$

于是,

$$K_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} K_{n-2};$$

利用此公式及

$$K_0 = x + C$$

即得

$$\begin{aligned} K_8 &= \int \cos^8 x dx = \frac{1}{8} \sin x \cos^7 x + \frac{7}{8} K_6 = \dots \\ &= \frac{1}{8} \sin x \cos^7 x + \frac{7}{48} \sin x \cos^5 x + \frac{35}{192} \sin x \cos^3 x \\ &\quad + \frac{35}{128} \sin x \cos x + \frac{35}{128} x + C. \end{aligned}$$

2012. 推出下列积分的递推公式

$$(a) \quad I_n = \int \frac{dx}{\sin^n x}; \quad (6) \quad K_n = \int \frac{dx}{\cos^n x} \quad (n > 2)$$

并利用推得的公式计算

$$\int \frac{dx}{\sin^5 x} \text{ 及 } \int \frac{dx}{\cos^7 x}.$$

$$\begin{aligned} \text{解} \quad (a) \quad I_n &= \int \frac{dx}{\sin^n x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^n x} dx \\ &= I_{n-2} - \frac{1}{n-1} \int \cos x d\left(\frac{1}{\sin^{n-1} x}\right) \\ &= I_{n-2} - \frac{\cos x}{(n-1)\sin^{n-1} x} - \frac{1}{n-1} I_{n-2} \\ &= -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}; \end{aligned}$$

利用此公式及

$$I_1 = \int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C,$$

即得

$$\begin{aligned} I_5 &= \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} + \frac{3}{4} I_3 = \dots \\ &= -\frac{\cos x}{4\sin^4 x} - \frac{3\cos x}{8\sin^2 x} + \frac{3}{8} \ln \left| \operatorname{tg} \frac{x}{2} \right| + C. \end{aligned}$$

$$\begin{aligned} (6) \quad K_n &= \int \frac{dx}{\cos^n x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^n x} dx \\ &= \frac{1}{n-1} \int \sin x d\left(\frac{1}{\cos^{n-1} x}\right) + K_{n-2} \\ &= \frac{\sin x}{(n-1)\cos^{n-1} x} - \frac{1}{n-1} K_{n-2} + K_{n-2} \\ &= \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} K_{n-2}; \end{aligned}$$

利用此公式及

$$K_1 = \int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C,$$

即得

$$\begin{aligned} K_7 &= \int \frac{dx}{\cos^7 x} = \frac{\sin x}{6\cos^6 x} + \frac{5}{6}K_5 = \dots \\ &= \frac{\sin x}{6\cos^6 x} + \frac{5\sin x}{24\cos^4 x} + \frac{5\sin x}{16\cos^2 x} \\ &\quad + \frac{5}{16} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C. \end{aligned}$$

运用公式

$$\text{I} \quad \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)];$$

$$\text{II} \quad \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)];$$

$$\text{III} \quad \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

来计算下列的积分.

求积分:

$$2013. \int \sin 5x \cos x dx.$$

$$\begin{aligned} \text{解} \quad \int \sin 5x \cos x dx &= \frac{1}{2} \int [\sin 4x + \sin 6x] dx \\ &= -\frac{1}{8} \cos 4x - \frac{1}{12} \cos 6x + C. \end{aligned}$$

$$2014. \int \cos x \cos 2x \cos 3x dx.$$

$$\begin{aligned} \text{解} \quad &\int \cos x \cos 2x \cos 3x dx \\ &= \frac{1}{2} \int \cos 2x (\cos 4x + \cos 2x) dx \\ &= \frac{1}{4} \int (\cos 6x + \cos 2x) dx + \frac{1}{4} \int (1 + \cos 4x) dx \end{aligned}$$

$$\begin{aligned}
\text{解} \quad & \int \cos^2 ax \cos^2 bx dx = \int (\cos ax \cos bx)^2 dx \\
&= \frac{1}{4} \int [\cos(a-b)x + \cos(a+b)x]^2 dx \\
&= \frac{1}{4} \int [\cos^2(a-b)x + \cos^2(a+b)x \\
&\quad + 2\cos(a-b)x \cos(a+b)x] dx \\
&= \frac{1}{8} \int [2 + \cos 2(a+b)x + \cos 2(a-b)x] dx \\
&\quad + \frac{1}{4} \int (\cos 2ax + \cos 2bx) dx \\
&= \frac{x}{4} + \frac{\sin 2(a+b)x}{16(a+b)} + \frac{\sin 2(a-b)x}{16(a-b)} \\
&\quad + \frac{1}{8a} \sin 2ax + \frac{1}{8b} \sin 2bx + C.
\end{aligned}$$

2018. $\int \sin^3 2x \cos^2 3x dx.$

解 先利用三角公式化简 $\sin^3 2x \cos^2 3x$, 得

$$\begin{aligned}
\sin^3 2x \cos^2 3x &= -\frac{1}{16} \sin 12x + \frac{3}{16} \sin 8x \\
&\quad - \frac{1}{8} \sin 6x - \frac{3}{16} \sin 4x + \frac{3}{8} \sin 2x,
\end{aligned}$$

于是

$$\begin{aligned}
& \int \sin^3 2x \cos^2 3x dx \\
&= \frac{1}{192} \cos 12x - \frac{3}{128} \cos 8x + \frac{1}{48} \cos 6x \\
&\quad + \frac{3}{64} \cos 4x - \frac{3}{16} \cos 2x + C.
\end{aligned}$$

运用恒等式

$$\sin(\alpha - \beta) = \sin[(x + \alpha) - (x + \beta)]$$

及 $\cos(\alpha - \beta) = \cos[(x + a) - (x + \beta)]$

来计算积分.

求积分:

$$2019. \int \frac{dx}{\sin(x+a)\sin(x+b)}.$$

$$\begin{aligned} \text{解} \quad & \int \frac{dx}{\sin(x+a)\sin(x+b)} \\ &= \frac{1}{\sin(a-b)} \int \frac{\sin[(a+x) - (x+b)]}{\sin(x+a)\sin(x+b)} dx \\ &= \frac{1}{\sin(a-b)} \int \left[\frac{\cos(x+b)}{\sin(x+b)} - \frac{\cos(x+a)}{\sin(x+a)} \right] dx \\ &= \frac{1}{\sin(a-b)} \ln \left| \frac{\sin(x+b)}{\sin(x+a)} \right| + C, \end{aligned}$$

其中设 $\sin(a-b) \neq 0$.

$$2020. \int \frac{dx}{\sin(x+a)\cos(x+b)}.$$

$$\begin{aligned} \text{解} \quad & \int \frac{dx}{\sin(x+a)\cos(x+b)} \\ &= \frac{1}{\cos(a-b)} \int \frac{\cos[(x+a) - (x+b)]}{\sin(x+a)\cos(x+b)} dx \\ &= \frac{1}{\cos(a-b)} \int \left[\frac{\cos(x+a)}{\sin(x+a)} + \frac{\sin(x+b)}{\cos(x+b)} \right] dx \\ &= \frac{1}{\cos(a-b)} \ln \left| \frac{\sin(x+a)}{\cos(x+b)} \right| + C, \end{aligned}$$

其中设 $\cos(a-b) \neq 0$.

$$2021. \int \frac{dx}{\cos(x+a)\cos(x+b)}.$$

$$\begin{aligned} \text{解} \quad & \int \frac{dx}{\cos(x+a)\cos(x+b)} \\ &= \frac{1}{\sin(a-b)} \int \frac{\sin[(x+a) - (x+b)]}{\cos(x+a)\cos(x+b)} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sin(a-b)} \int \left(\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)} \right) dx \\
&= \frac{1}{\sin(a-b)} \ln \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C,
\end{aligned}$$

其中设 $\sin(a-b) \neq 0^*$).

*) 当 $a-b=2k\pi (k=0, \pm 1, \pm 2, \dots)$ 时, 是更简单的积分, 2019 题及 2020 题与本题类似, 解法从略.

2022. $\int \frac{dx}{\sin x - \sin a}.$

解
$$\begin{aligned}
\int \frac{dx}{\sin x - \sin a} &= \frac{1}{\cos a} \int \frac{\cos \left(\frac{x+a}{2} - \frac{x-a}{2} \right)}{\sin x - \sin a} dx \\
&= \frac{1}{\cos a} \int \frac{\cos \frac{x+a}{2} \cos \frac{x-a}{2} + \sin \frac{x+a}{2} \sin \frac{x-a}{2}}{2 \cos \frac{x+a}{2} \sin \frac{x-a}{2}} dx \\
&= \frac{1}{2 \cos a} \int \left(\frac{\cos \frac{x-a}{2}}{\sin \frac{x-a}{2}} + \frac{\sin \frac{x+a}{2}}{\cos \frac{x+a}{2}} \right) dx \\
&= \frac{1}{\cos a} \ln \left| \frac{\sin \frac{x-a}{2}}{\cos \frac{x+a}{2}} \right| + C,
\end{aligned}$$

其中设 $\cos a \neq 0$.

2023. $\int \frac{dx}{\cos x + \cos a}.$

解
$$\int \frac{dx}{\cos x + \cos a} = \int \frac{d \left(x + \frac{\pi}{2} \right)}{\sin \left(x + \frac{\pi}{2} \right) - \sin \left(a + \frac{3}{2} \pi \right)}$$

$$\begin{aligned}
&= \frac{1}{\cos\left(a + \frac{3}{2}\pi\right)} \ln \left| \frac{\sin \frac{x-a-\pi}{2}}{\cos \frac{x+a+2\pi}{2}} \right|^{*}) + C \\
&= \frac{1}{\sin a} \ln \left| \frac{\cos \frac{x-a}{2}}{\cos \frac{x+a}{2}} \right| + C,
\end{aligned}$$

其中设 $\sin a \neq 0$.

*) 利用 2022 题的结果.

2024. $\int \operatorname{tg} x \operatorname{tg}(x+a) dx.$

$$\begin{aligned}
&\text{解 } \int \operatorname{tg} x \operatorname{tg}(x+a) dx \\
&= \int \frac{\sin x \sin(x+a)}{\cos x \cos(x+a)} dx \\
&= \int \frac{\cos x \cos(x+a) + \sin x \sin(x+a) - \cos x \cos(x+a)}{\cos x \cos(x+a)} dx \\
&= \int \frac{\cos a - \cos x \cos(x+a)}{\cos x \cos(x+a)} dx \\
&= -x + \cos a \cdot \int \frac{dx}{\cos(x+a) \cos x} \\
&= -x + \operatorname{ctg} a \cdot \ln \left| \frac{\cos x}{\cos(x+a)} \right|^{*}) + C,
\end{aligned}$$

其中设 $\sin a \neq 0$.

*) 利用 2021 题的结果.

形如

$$\int R(\sin x, \cos x) dx$$

(式中 R 为有理函数) 的积分的一般情形可利用代换

$\operatorname{tg} \frac{x}{2} = t$ 化为有理函数的积分.

(a) 若等式

$$R(-\sin x, \cos x) \equiv -R(\sin x, \cos x),$$

或 $R(\sin x, -\cos x) \equiv -R(\sin x, \cos x)$

成立, 则最好利用代换 $\cos x = t$ 或对应的 $\sin x = t$.

(6) 若等式

$$R(-\sin x, -\cos x) \equiv R(\sin x, \cos x)$$

成立, 则最好利用代换 $\operatorname{tg} x = t$.

求积分:

$$2025. \int \frac{dx}{2\sin x - \cos x + 5}.$$

解 设 $t = \operatorname{tg} \frac{x}{2}$, 则 $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$,

$$dx = \frac{2dt}{1+t^2}.$$

于是,

$$\begin{aligned} \int \frac{dx}{2\sin x - \cos x + 5} &= \int \frac{dt}{3t^2 + 2t + 2} \\ &= \frac{1}{\sqrt{5}} \operatorname{arctg} \left(\frac{3t+1}{\sqrt{5}} \right) + C \\ &= \frac{1}{\sqrt{5}} \operatorname{arctg} \left[\frac{3\operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} \right] + C. \end{aligned}$$

$$2026. \int \frac{dx}{(2+\cos x)\sin x}.$$

解 设 $t = \operatorname{tg} \frac{x}{2}$. 同 2025 题, 得

$$\begin{aligned} \int \frac{dx}{(2+\cos x)\sin x} &= \int \frac{1+t^2}{t(3+t^2)} dt \\ &= \int \left(\frac{1}{3t} + \frac{2t}{3(3+t^2)} \right) dt \end{aligned}$$

$$= \frac{1}{3} \ln |t(3+t^2)| + C_1$$

$$= \frac{1}{6} \ln \frac{(1-\cos x)(2+\cos x)^2}{(1+\cos x)^3} + C.$$

*) 由于

$$\begin{aligned} t(3+t^2) &= \operatorname{tg} \frac{x}{2} \left(2 + \sec^2 \frac{x}{2} \right) \\ &= \frac{\sin \frac{x}{2}}{\cos^3 \frac{x}{2}} \left(1 + 2\cos^2 \frac{x}{2} \right) \\ &= \frac{\left(\frac{1-\cos x}{2} \right)^{\frac{1}{2}}}{\left(\frac{1+\cos x}{2} \right)^{\frac{3}{2}}} (\cos x + 2) \\ &= 2 \left[\frac{(1-\cos x)(\cos x + 2)^2}{(1+\cos x)^3} \right]^{\frac{1}{2}}, \end{aligned}$$

因而

$$\ln |t(3+t^2)| = \ln 2 + \frac{1}{2} \ln \frac{(1-\cos x)(2+\cos x)^2}{(1+\cos x)^3}.$$

2027. $\int \frac{\sin^2 x}{\sin x + 2\cos x} dx.$

解 设 $\operatorname{tg} \frac{x}{2} = t$, 同 2025 题, 得

$$\begin{aligned} \int \frac{\sin^2 x}{\sin x + 2\cos x} dx &= 4 \int \frac{t^2 dt}{(1+t^2)^2(1+t-t^2)} \\ &= \frac{4}{5} \int \left(\frac{1}{1+t^2} + \frac{-2+t}{(1+t^2)^2} + \frac{1}{1+t-t^2} \right) dt \\ &= \frac{4}{5} \int \frac{dt}{1+t^2} - \frac{8}{5} \int \frac{dt}{(1+t^2)^2} + \frac{2}{5} \int \frac{2t dt}{(1+t^2)^2} \end{aligned}$$

$$\begin{aligned}
& + \frac{4}{5} \int \frac{d\left(t - \frac{1}{2}\right)}{\frac{5}{4} - \left(t - \frac{1}{2}\right)^2} \\
& = \frac{4}{5} \arctg t - \frac{8}{5} \left\{ \frac{t}{2(1+t^2)} + \frac{1}{2} \arctg t \right\}^{*}) \\
& \quad - \frac{2}{5} \cdot \frac{1}{1+t^2} + \frac{4}{5\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}}{2} + t - \frac{1}{2}}{\frac{\sqrt{5}}{2} - (t - \frac{1}{2})} \right| + C_1 \\
& = -\frac{2}{5} \cdot \frac{1+2t}{1+t^2} + \frac{4}{5\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}-1}{2} + t}{\frac{\sqrt{5}+1}{2} - t} \right| + C_1 \\
& = -\frac{1}{5} (\cos x + 2\sin x)^{**}) \\
& \quad + \frac{4}{5\sqrt{5}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\operatorname{arctg} 2}{2} \right) \right|^{***}) + C.
\end{aligned}$$

*) 利用 1817 题的结果.

$$**) \quad -\frac{2}{5} \cdot \frac{1+2t}{1+t^2} = -\frac{2}{5} \cdot \frac{1+2\operatorname{tg} \frac{x}{2}}{\sec^2 \frac{x}{2}}$$

$$= -\frac{2}{5} \cdot \frac{1+2 \cdot \frac{\sin x}{1+\cos x}}{2}$$

$$= -\frac{1}{5} (\cos x + 2\sin x) - \frac{1}{5}.$$

$$\begin{aligned}
& \text{***}) \ln \left| \frac{\frac{\sqrt{5}-1}{2} + t}{\frac{\sqrt{5}+1}{2} - t} \right| \\
&= \ln \left| \frac{\operatorname{tg}\left(\frac{\operatorname{arctg} 2}{2}\right) + \operatorname{tg} \frac{x}{2}}{\operatorname{ctg}\left(\frac{\operatorname{arctg} 2}{2}\right) - \operatorname{tg} \frac{x}{2}} \right| \\
&= \ln \left| \frac{\operatorname{tg}\left(\frac{\operatorname{arctg} 2}{2}\right) + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg}\left(\frac{\operatorname{arctg} 2}{2}\right) \cdot \operatorname{tg} \frac{x}{2}} \right| + \ln \frac{1}{\operatorname{ctg}\left(\frac{\operatorname{arctg} 2}{2}\right)} \\
&= \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\operatorname{arctg} 2}{2}\right) \right| - \ln \left(\operatorname{ctg}\left(\frac{\operatorname{arctg} 2}{2}\right) \right).
\end{aligned}$$

2028. $\int \frac{dx}{1 + \epsilon \cos x};$

(a) $0 < \epsilon < 1$; (б) $\epsilon > 1$.

解 设 $t = \operatorname{tg} \frac{x}{2}$. 同 2025 题, 得

$$\int \frac{dx}{1 + \epsilon \cos x} = 2 \int \frac{dt}{(1 + \epsilon) + (1 - \epsilon)t^2} = I.$$

(a) $0 < \epsilon < 1$,

$$\begin{aligned}
I &= \frac{2}{1 + \epsilon} \int \frac{dt}{1 + \left(\frac{1 - \epsilon}{1 + \epsilon}\right)t^2} \\
&= \frac{2}{\sqrt{1 - \epsilon^2}} \operatorname{arctg} \left[t \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \right] + C \\
&= \frac{2}{\sqrt{1 - \epsilon^2}} \operatorname{arctg} \left[\sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \operatorname{tg} \frac{x}{2} \right] + C;
\end{aligned}$$

(б) $\epsilon > 1$,

$$\begin{aligned}
 I &= \frac{2}{\epsilon - 1} \int \frac{dt}{\left(\frac{\epsilon + 1}{\epsilon - 1}\right) - t^2} \\
 &= \frac{1}{\sqrt{\epsilon^2 - 1}} \ln \left| \frac{\sqrt{\epsilon + 1} + \sqrt{\epsilon - 1}t}{\sqrt{\epsilon + 1} - \sqrt{\epsilon - 1}t} \right| + C \\
 &= \frac{1}{\sqrt{\epsilon^2 - 1}} \ln \left| \frac{\epsilon + \cos x + \sqrt{\epsilon^2 - 1} \sin x}{1 + \epsilon \cos x} \right|^{*}) \\
 &\quad + C.
 \end{aligned}$$

$$\begin{aligned}
 *) \quad & \frac{\sqrt{\epsilon + 1} + t \sqrt{\epsilon - 1}}{\sqrt{\epsilon + 1} - t \sqrt{\epsilon - 1}} \\
 &= \frac{\epsilon + 1 + 2t \sqrt{\epsilon^2 - 1} + (\epsilon - 1)t^2}{(\epsilon + 1) - (\epsilon - 1)t^2} \\
 &= \frac{\epsilon(1 + t^2) + (1 - t^2) + 2 \sqrt{\epsilon^2 - 1}t}{\epsilon(1 - t^2) + (1 + t^2)} \\
 &= \frac{\epsilon(1 + t^2) + (1 + t^2)\cos x + 2t \sqrt{\epsilon^2 - 1}}{\epsilon(1 + t^2)\cos x + (1 + t^2)} \\
 &= \frac{\epsilon + \cos x + \sqrt{\epsilon^2 - 1} \cdot \frac{2t}{1 + t^2}}{\epsilon \cos x + 1} \\
 &= \frac{\epsilon + \cos x + \sqrt{\epsilon^2 - 1} \sin x}{\epsilon \cos x + 1}.
 \end{aligned}$$

2029. $\int \frac{\sin^2 x}{1 + \sin^2 x} dx.$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\sin^2 x}{1 + \sin^2 x} dx = \int \left(1 - \frac{1}{1 + \sin^2 x} \right) dx \\
 &= x - \int \frac{d(\operatorname{tg} x)}{\sec^2 x + \operatorname{tg}^2 x} = x - \int \frac{d(\operatorname{tg} x)}{1 + 2\operatorname{tg}^2 x} \\
 &= x - \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg}(\sqrt{2} \operatorname{tg} x) + C.
 \end{aligned}$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{8} \right) \right| + C.$$

$$2033. \int \frac{dx}{(a \sin x + b \cos x)^2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{(a \sin x + b \cos x)^2} &= \frac{1}{a} \int \frac{d(atgx + b)}{(atgx + b)^2} \\ &= -\frac{1}{atgx + b} + C = -\frac{\cos x}{a(a \sin x + b \cos x)} + C. \end{aligned}$$

$$2034. \int \frac{\sin x dx}{\sin^3 x + \cos^3 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin x dx}{\sin^3 x + \cos^3 x} &= \int \frac{\sin x dx}{(\sin x + \cos x)(1 - \sin x \cos x)} \\ &= \frac{1}{2} \int \frac{(\sin x - \cos x) dx}{(\sin x + \cos x)(1 - \sin x \cos x)} \\ &\quad + \frac{1}{2} \int \frac{dx}{1 - \sin x \cos x} \\ &= \frac{1}{3} \int \frac{-(\cos x - \sin x) dx}{\sin x + \cos x} \\ &\quad + \frac{1}{6} \int \frac{\sin^2 x - \cos^2 x}{1 - \sin x \cos x} dx + \frac{1}{2} \int \frac{dx}{1 - \sin x \cos x} \\ &= -\frac{1}{3} \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} + \frac{1}{6} \int \frac{d(1 - \sin x \cos x)}{1 - \sin x \cos x} \\ &\quad - \frac{1}{\sqrt{3}} \int d \left(\operatorname{arc} \operatorname{tg} \frac{2 \cos x - \sin x}{\sqrt{3} \sin x} \right) \\ &= -\frac{1}{6} \ln \frac{(\sin x + \cos x)^2}{1 - \sin x \cos x} \\ &\quad - \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2 \cos x - \sin x}{\sqrt{3} \sin x} \right) + C. \end{aligned}$$

$$2035. \int \frac{dx}{\sin^4 x + \cos^4 x}.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{dx}{\sin^4 x + \cos^4 x} = \int \frac{2dx}{2 - \sin^2 2x} \\
 &= \int \frac{d(\operatorname{tg} 2x)}{2 \sec^2 2x - \operatorname{tg}^2 2x} \\
 &= \int \frac{d(\operatorname{tg} 2x)}{2 + \operatorname{tg}^2 2x} = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} 2x}{\sqrt{2}} \right) + C.
 \end{aligned}$$

$$2036. \int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x} dx.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x} dx = \int \frac{2 \sin^2 2x dx}{\sin^4 2x - 8 \sin^2 2x + 8} \\
 &= \int \frac{\operatorname{tg}^2 2x d(\operatorname{tg} 2x)}{\operatorname{tg}^4 2x - 8 \operatorname{tg}^2 2x \sec^2 2x + 8 \sec^4 2x} \\
 &= \int \frac{\operatorname{tg}^2 2x d(\operatorname{tg} 2x)}{\operatorname{tg}^4 2x + 8 \operatorname{tg}^2 2x + 8} \\
 &= \frac{\sqrt{2}}{4} (2 + \sqrt{2}) \int \frac{d(\operatorname{tg} 2x)}{\operatorname{tg}^2 2x + 4 + 2\sqrt{2}} \\
 &\quad - \frac{\sqrt{2}}{4} (2 - \sqrt{2}) \int \frac{d(\operatorname{tg} 2x)}{\operatorname{tg}^2 2x + 4 - \sqrt{2}} \\
 &= \frac{1}{4} \left[\sqrt{2 + \sqrt{2}} \operatorname{arc} \operatorname{tg} \left[\frac{\operatorname{tg} 2x}{\sqrt{4 + 2\sqrt{2}}} \right] \right. \\
 &\quad \left. - \sqrt{2 - \sqrt{2}} \operatorname{arc} \operatorname{tg} \left[\frac{\operatorname{tg} 2x}{\sqrt{4 - 2\sqrt{2}}} \right] \right] + C.
 \end{aligned}$$

$$2037. \int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx = - \int \frac{\cos 2x}{1 - \frac{1}{2} \sin^2 2x} dx \\
 &= - \frac{1}{2\sqrt{2}} \int \left(\frac{2 \cos 2x}{\sqrt{2} - \sin 2x} + \frac{2 \cos 2x}{\sqrt{2} + \sin 2x} \right) dx
 \end{aligned}$$

$$= \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2} - \sin 2x}{\sqrt{2} + \sin 2x} + C.$$

2038. $\int \frac{\sin x \cos x}{1 + \sin^4 x} dx.$

解 $\int \frac{\sin x \cos x}{1 + \sin^4 x} dx = \int \frac{\operatorname{tg} x \sec^2 x}{\sec^4 x + \operatorname{tg}^4 x} dx$
 $= \frac{1}{2} \int \frac{d(\operatorname{tg}^2 x)}{2\operatorname{tg}^4 x + 2\operatorname{tg}^2 x + 1} = \frac{1}{2} \operatorname{arctg}(1 + 2\operatorname{tg}^2 x)$
 $+ C.$

2039. $\int \frac{dx}{\sin^6 x + \cos^6 x}.$

解 $\int \frac{dx}{\sin^6 x + \cos^6 x} = \int \frac{dx}{1 - 3\sin^2 x \cos^2 x}$
 $= \int \frac{dx}{1 - \frac{3}{4}\sin^2 2x} = \int \frac{2d(\operatorname{tg} 2x)}{4\sec^2 2x - 3\operatorname{tg}^2 2x}$
 $= \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} 2x}{2} \right) + C.$

2040. $\int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2}.$

解 $\int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2} = \int \frac{\frac{1}{\cos^4 x} dx}{(\operatorname{tg}^2 x + 2)^2}$
 $= \int \frac{\sec^2 x d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2}$
 $= \int \frac{\operatorname{tg}^2 x}{(\operatorname{tg}^2 x + 2)^2} d(\operatorname{tg} x) + \int \frac{d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2}$
 $= \int \frac{(\operatorname{tg}^2 x + 2) - 2}{(\operatorname{tg}^2 x + 2)^2} d(\operatorname{tg} x) + \int \frac{d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2}$
 $= \int \frac{d(\operatorname{tg} x)}{\operatorname{tg}^2 x + 2} - \int \frac{d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2}$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \arctan \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right) - \frac{\operatorname{tg} x}{4(\operatorname{tg}^2 x + 2)} \\
&\quad - \frac{1}{4\sqrt{2}} \arctan \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right)' + C \\
&= \frac{3}{4\sqrt{2}} \arctan \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right) - \frac{\operatorname{tg} x}{4(\operatorname{tg}^2 x + 2)} + C.
\end{aligned}$$

*) 利用 1817 题的结果.

2041. 求积分

$$\int \frac{dx}{a \sin x + b \cos x}$$

先化分母为对数的形状.

$$\begin{aligned}
\text{解} \quad \int \frac{dx}{a \sin x + b \cos x} &= \frac{1}{\sqrt{a^2 + b^2}} \int \frac{dx}{\sin(x + \varphi)} \\
&= \frac{1}{\sqrt{a^2 + b^2}} \ln \left| \operatorname{tg} \left(\frac{x + \varphi}{2} \right) \right| + C,
\end{aligned}$$

$$\text{其中 } \cos \varphi = \frac{a}{\sqrt{a^2 + b^2}}, \sin \varphi = \frac{b}{\sqrt{a^2 + b^2}},$$

并设 $a^2 + b^2 \neq 0$.

2042. 证明

$$\begin{aligned}
&\int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} dx \\
&= Ax + B \ln |a \sin x + b \cos x| + C.
\end{aligned}$$

式中 A, B, C 为常数.

$$\begin{aligned}
\text{证} \quad a_1 \sin x + b_1 \cos x &= A(a \sin x + b \cos x) \\
&\quad + B(a \cos x - b \sin x),
\end{aligned}$$

$$\text{式中 } A = \frac{aa_1 + bb_1}{a^2 + b^2}, B = \frac{ab_1 - a_1b}{a^2 + b^2}, a^2 + b^2 \neq 0.$$

于是

$$\begin{aligned}
& \int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} dx \\
&= A \int dx + B \int \frac{d(a \sin x + b \cos x)}{a \sin x + b \cos x} \\
&= Ax + B \ln |a \sin x + b \cos x| + C.
\end{aligned}$$

求积分:

2043. $\int \frac{\sin x - \cos x}{\sin x + 2 \cos x} dx.$

解 此为 2042 题的特例, 这里

$$a_1 = 1, b_1 = -1, a = 1, b = 2;$$

$$A = \frac{aa_1 + bb_1}{a^2 + b^2} = \frac{1 - 2}{1 + 4} = -\frac{1}{5},$$

$$B = \frac{ab_1 - a_1b}{a^2 + b^2} = \frac{-1 - 2}{1 + 4} = -\frac{3}{5}.$$

代入得

$$\begin{aligned}
& \int \frac{\sin x - \cos x}{\sin x + 2 \cos x} dx \\
&= -\frac{x}{5} - \frac{3}{5} \ln |\sin x + 2 \cos x| + C.
\end{aligned}$$

2044. $\int \frac{dx}{3 + 5 \operatorname{tg} x}.$

解 $\int \frac{dx}{3 + 5 \operatorname{tg} x} = \int \frac{\cos x}{5 \sin x + 3 \cos x} dx.$

此为 2042 题的特例, 这里

$$a_1 = 0, b_1 = 1, a = 5, b = 3;$$

$$A = \frac{3}{34}, B = \frac{5}{34}.$$

代入得

$$\int \frac{dx}{3 + 5 \operatorname{tg} x} = \frac{3}{34} x + \frac{5}{34} \ln |5 \sin x + 3 \cos x| + C.$$

$$2045. \int \frac{a_1 \sin x + b_1 \cos x}{(a \sin x + b \cos x)^2} dx.$$

$$\begin{aligned} \text{解} \quad & \text{仿 2042 题, } \int \frac{a_1 \sin x + b_1 \cos x}{(a \sin x + b \cos x)^2} dx \\ &= A \int \frac{a \sin x + b \cos x}{(a \sin x + b \cos x)^2} dx \\ &+ B \int \frac{a \cos x - b \sin x}{(a \sin x + b \cos x)^2} dx \\ &= A \int \frac{dx}{a \sin x + b \cos x} + B \int \frac{d(a \sin x + b \cos x)}{(a \sin x + b \cos x)^2} dx \\ &= \frac{A}{\sqrt{a^2 + b^2}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\varphi}{2} \right) \right|^{*}) \\ &- \frac{B}{a \sin x + b \cos x} + C \\ &= \frac{aa_1 + bb_1}{(a^2 + b^2)^{\frac{3}{2}}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\varphi}{2} \right) \right| \\ &- \frac{ab_1 - a_1 b}{(a^2 + b^2)(a \sin x + b \cos x)} + C, \end{aligned}$$

$$\text{式中 } A = \frac{aa_1 + bb_1}{a^2 + b^2}, B = \frac{ab_1 - a_1 b}{a^2 + b^2},$$

$$\cos \varphi = \frac{a}{\sqrt{a^2 + b^2}}, \sin \varphi = \frac{b}{\sqrt{a^2 + b^2}},$$

$a^2 + b^2 \neq 0$ (显然按题意 a, b 不同时为零).

*) 利用 2041 题的结果.

2046. 证明:

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x + c_1}{a \sin x + b \cos x + c} dx \\ &= Ax + B \ln |a \sin x + b \cos x + c| \\ &+ C \int \frac{dx}{a \sin x + b \cos x + c}, \end{aligned}$$

式中 A, B, C 都是常系数.

证 按题意 a, b 不同时为零. 设

$$a_1 \sin x + b_1 \cos x + c_1 \equiv A(a \sin x + b \cos x + c) \\ + B(a \cos x - b \sin x) + C,$$

比较等式两端同类项的系数, 则有

$$A = \frac{aa_1 + bb_1}{a^2 + b^2}, B = \frac{ab_1 - a_1b}{a^2 + b^2}, \\ C = \frac{a(ac_1 - a_1c) + b(bc_1 - b_1c)}{a^2 + b^2}.$$

代入得

$$\int \frac{a_1 \sin x + b_1 \cos x + c_1}{a \sin x + b \cos x + c} dx \\ = A \int dx + B \int \frac{d(a \sin x + b \cos x + c)}{a \sin x + b \cos x + c} \\ + C \int \frac{dx}{a \sin x + b \cos x + c} \\ = Ax + B \ln |a \sin x + b \cos x + c| \\ + C \int \frac{dx}{a \sin x + b \cos x + c}.$$

求积分:

$$2047. \int \frac{\sin x + 2 \cos x - 3}{\sin x - 2 \cos x + 3} dx.$$

解 此为 2046 题之特例, 这里

$$a_1 = 1, b_1 = 2, c_1 = -3, a = 1, b = -2, c = 3;$$

$$A = \frac{aa_1 + bb_1}{a^2 + b^2} = \frac{1 - 4}{1 + 4} = -\frac{3}{5},$$

$$B = \frac{ab_1 - a_1b}{a^2 + b^2} = \frac{2 + 2}{1 + 4} = \frac{4}{5},$$

$$C = \frac{a(ac_1 - a_1c) + b(bc_1 - b_1c)}{a^2 + b^2}$$

$$= \frac{(-3-3) + (-2)(6-6)}{1+4} = -\frac{6}{5}.$$

代入得

$$\begin{aligned} & \int \frac{\sin x + 2\cos x - 3}{\sin x - 2\cos x + 3} dx \\ &= -\frac{3}{5}x + \frac{4}{5}\ln|\sin x - 2\cos x + 3| \\ & \quad - \frac{6}{5} \int \frac{dx}{\sin x - 2\cos x + 3} \\ &= -\frac{3}{5}x + \frac{4}{5}\ln|\sin x - 2\cos x + 3| \\ & \quad - \frac{6}{5} \operatorname{arc\,tg} \frac{1 + 5\operatorname{tg} \frac{x}{2}}{2} + C. \end{aligned}$$

*) 设 $t = \operatorname{tg} \frac{x}{2}$, 积分即得所求式子.

2048. $\int \frac{\sin x dx}{\sqrt{2} + \sin x + \cos x}.$

解 此为 2046 题之特例, 这里

$$a_1 = 1, b_1 = 0, c_1 = 0, a = 1, b = 1, c = \sqrt{2};$$

$$A = \frac{1}{2}, B = -\frac{1}{2}, C = -\frac{1}{\sqrt{2}}.$$

代入得

$$\begin{aligned} & \int \frac{\sin x dx}{\sqrt{2} + \sin x + \cos x} \\ &= \frac{1}{2}x - \frac{1}{2}\ln|\sqrt{2} + \sin x + \cos x| \\ & \quad - \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{2} + \sin x + \cos x} \\ &= \frac{1}{2}x - \frac{1}{2}\ln|\sqrt{2} + \sin x + \cos x| \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{2} + \sqrt{2} \cos\left(x - \frac{\pi}{4}\right)} \\
& = \frac{1}{2}x - \frac{1}{2} \ln |\sqrt{2} + \sin x + \cos x| \\
& - \frac{1}{2} \int \frac{dx}{2 \cos^2\left(\frac{x}{2} - \frac{\pi}{8}\right)} \\
& = \frac{1}{2}x - \frac{1}{2} \ln |\sqrt{2} + \sin x + \cos x| \\
& - \frac{1}{2} \operatorname{tg}\left(\frac{x}{2} - \frac{\pi}{8}\right) + C.
\end{aligned}$$

2049. $\int \frac{2\sin x + \cos x}{3\sin x + 4\cos x - 2} dx.$

解 本题也是 2046 题之特例, 这里

$$a_1 = 2, b_1 = 1, c_1 = 0, a = 3, b = 4, c = -2;$$

$$A = \frac{2}{5}, B = -\frac{1}{5}, C = \frac{4}{5}.$$

代入得

$$\begin{aligned}
& \int \frac{2\sin x + \cos x}{3\sin x + 4\cos x - 2} dx \\
& = \frac{2}{5}x - \frac{1}{5} \ln |3\sin x + 4\cos x - 2| \\
& + \frac{4}{5} \int \frac{dx}{3\sin x + 4\cos x - 2} \\
& = \frac{2}{5}x - \frac{1}{5} \ln |3\sin x + 4\cos x - 2| \\
& + \frac{4}{5\sqrt{21}} \ln \left| \frac{\sqrt{7} + \sqrt{3} \left(2\operatorname{tg} \frac{x}{2} - 1\right)}{\sqrt{7} - \sqrt{3} \left(2\operatorname{tg} \frac{x}{2} - 1\right)} \right|^{*}) \\
& + C.
\end{aligned}$$

解 此为 2050 题之特例, 这里

$$a_1 = 1, b_1 = -2, c_1 = 3, a = 1, b = 1;$$

$$A = \frac{bc_1 - a_1b + 2ab_1}{a^2 + b^2} = \frac{3 - 1 - 4}{1 + 1} = -1,$$

$$B = \frac{ac_1 - aa_1 - 2bb_1}{a^2 + b^2} = \frac{3 - 1 + 4}{1 + 1} = 3,$$

$$C = \frac{a_1b^2 + a^2c_1 - 2abb_1}{a^2 + b^2} = \frac{1 + 3 + 4}{1 + 1} = 4.$$

代入得

$$\begin{aligned} & \int \frac{\sin^2 x - 4\sin x \cos x + 3\cos^2 x}{\sin x + \cos x} dx \\ &= -\sin x + 3\cos x + 4 \int \frac{dx}{\sin x + \cos x} \\ &= -\sin x + 3\cos x + \frac{4}{\sqrt{2}} \int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} \\ &= -\sin x + 3\cos x + 2\sqrt{2} \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{8}\right) \right| + C. \end{aligned}$$

2052. $\int \frac{\sin^2 x - \sin x \cos x + 2\cos^2 x}{\sin x + 2\cos x} dx.$

解 本题也是 2050 题的特例, 这里

$$a_1 = 1, b_1 = -\frac{1}{2}, c_1 = 2, a = 1, b = 2;$$

$$A = \frac{1}{5}, B = \frac{3}{5}, C = \frac{8}{5}$$

代入得

$$\begin{aligned} & \int \frac{\sin^2 x - \sin x \cos x + 2\cos^2 x}{\sin x + 2\cos x} dx \\ &= \frac{1}{5} \sin x + \frac{3}{5} \cos x + \frac{8}{5} \int \frac{dx}{\sin x + 2\cos x} \end{aligned}$$

$$= \frac{1}{5}(\sin x + 3\cos x).$$

$$+ \frac{8}{5\sqrt{5}} \ln \left| \frac{\sqrt{5} + 2\operatorname{tg} \frac{x}{2} - 1}{\sqrt{5} - 2\operatorname{tg} \frac{x}{2} + 1} \right|^{*}) + C.$$

*) 设 $t = \operatorname{tg} \frac{x}{2}$, 积分即得所求式子.

2053. 证明: 若 $(a - c)^2 + b^2 \neq 0$, 则

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x}{a \sin^2 x + 2b \sin x \cos x + c \cos^2 x} dx \\ &= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2}, \end{aligned}$$

式中 A, B 为未定系数, λ_1, λ_2 为下方程式的根

$$\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0 \quad (\lambda_1 \neq \lambda_2)$$

及

$$u_i = (a - \lambda_i) \sin x + b \cos x, k_i = \frac{1}{a - \lambda_i} (i = 1, 2).$$

证 记

$$\begin{aligned} & a^2 \sin^2 x + 2b \sin x \cos x + c \cos^2 x \\ &= (a - \lambda_i) \sin^2 x + 2b \sin x \cos x \\ &+ (c - \lambda_i) \cos^2 x + \lambda_i \\ &= \frac{1}{a - \lambda_i} [(a - \lambda_i)^2 \sin^2 x + 2b(a - \lambda_i) \sin x \cos x \\ &+ (c - \lambda_i)(a - \lambda_i) \cos^2 x] + \lambda_i, \end{aligned}$$

其中 $\lambda_i (i = 1, 2)$ 为 $\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0$ 的根.

由假定 $(a - c)^2 + b^2 \neq 0$, 从而 $(a - c)^2 + 4b^2 \neq 0$,

因此 $\lambda_1 \neq \lambda_2$.

再设 $k_i = \frac{1}{a - \lambda_i} (i = 1, 2)$ 及

$$u_i = (a - \lambda_i)\sin x + b\cos x.$$

由于 $(a - \lambda_1)(c - \lambda_1) - b^2 = 0$, 即 $b^2 = (a - \lambda_1)(c - \lambda_1)$.
于是,

$$\begin{aligned} & a^2\sin^2x + 2b\sin x\cos x + c\cos^2x \\ &= k_i[(a - \lambda_i)^2\sin^2x + 2b(a - \lambda_i)\sin x\cos x \\ & \quad + b^2\cos^2x] + \lambda_i = k_i[(a - \lambda_i)\sin x + b\cos x]^2 \\ & \quad + \lambda_i = k_i u_i^2 + \lambda_i. \end{aligned} \quad (1)$$

其次, 设

$$\begin{aligned} a_1\sin x + b_1\cos x &= A[(a - \lambda_1)\cos x - b\sin x] \\ & \quad + B[(a - \lambda_2)\cos x - b\sin x], \end{aligned} \quad (2)$$

比较等式两端同类项的系数, 则有

$$\begin{aligned} -b(A + B) &= a_1, \\ A(a - \lambda_1) + B(a - \lambda_2) &= b_1, \\ A &= -\frac{a_1(\lambda_1 - \lambda_2) + bb_1 + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)}, \\ B &= \frac{bb_1 + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)}. \end{aligned}$$

由(1)式及(2)式即得

$$\begin{aligned} & \int \frac{a_1\sin x + b_1\cos x}{a^2\sin^2x + 2b\sin x\cos x + c\cos^2x} dx \\ &= A \int \frac{(a - \lambda_1)\cos x - b\sin x}{k_1[(a - \lambda_1)\sin x + b\cos x]^2 + \lambda_1} dx \\ & \quad + B \int \frac{(a - \lambda_2)\cos x - b\sin x}{k_2[(a - \lambda_2)\sin x + b\cos x]^2 + \lambda_2} dx \end{aligned}$$

$$= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2}.$$

*) 按题意, $b \neq 0$. 因若 $b = 0$, 则 $\lambda_1 = a, \lambda_2 = c$, 从而 k_i 无意义. 不过, 当 $b = 0$ 时, 仍能化为所要求的类似形式. 事实上, 当 $b = 0$ 时, $a \neq c$,

我们有

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x}{a \sin^2 x + 2b \sin x \cos x + c \cos^2 x} dx \\ &= \int \frac{a_1 \sin x + b_1 \cos x}{a \sin^2 x + c \cos^2 x} dx \\ &= a_1 \int \frac{\sin x}{a \sin^2 x + c \cos^2 x} dx + b_1 \int \frac{\cos x}{a \sin^2 x + c \cos^2 x} dx \\ &= a_1 \int \frac{d(\cos x)}{(c - a) \cos^2 x + a} + b_1 \int \frac{d(\sin x)}{(a - c) \sin^2 x + c} \\ &= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2}, \end{aligned}$$

式中 $A = -a_1, B = b_1, k_1 = c - a, k_2 = a - c$,

$$u_1 = \cos x, u_2 = \sin x, \lambda_1 = a, \lambda_2 = c.$$

本题也可用下法另证: 命 $u_i = (a - \lambda_i) \sin x + b \cos x$,

$k_i = \frac{1}{a - \lambda_i} (i = 1, 2)$, 代入积分等式. 然后两边求导,

整理并比较系数, 便可知 λ_i 必为 $\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0$

的根, 相应可求出系数, A, B .

求积分:

$$2054. \int \frac{2 \sin x - \cos x}{3 \sin^2 x + 4 \cos^2 x} dx.$$

$$\text{解} \quad \int \frac{2 \sin x - \cos x}{3 \sin^2 x + 4 \cos^2 x} dx$$

$$\begin{aligned}
&= \int \frac{2\sin x}{3\sin^2 x + 4\cos^2 x} dx - \int \frac{\cos x}{3\sin^2 x + 4\cos^2 x} dx \\
&= -2 \int \frac{d(\cos x)}{3 + \cos^2 x} - \int \frac{d(\sin x)}{4 - \sin^2 x} \\
&= -\frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{\cos x}{\sqrt{3}}\right) - \frac{1}{4} \ln \frac{2 + \sin x}{2 - \sin x} + C.
\end{aligned}$$

2055. $\int \frac{\sin x + \cos x}{2\sin^2 x - 4\sin x \cos x + 5\cos^2 x} dx.$

解 此为 2053 题之特例, 这里

$$a_1 = 1, b_1 = 1, a = 2, b = -2, c = 5.$$

由

$$\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0,$$

求得 $\lambda_1 = 1, \lambda_2 = 6$. 从而

$$\begin{aligned}
A &= -\frac{a_1(\lambda_1 - \lambda_2) + b_1 b + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)} \\
&= \frac{(1 - 6) - 2 + (2 - 1)}{-2(1 - 6)} = \frac{3}{5},
\end{aligned}$$

$$B = \frac{bb_1 + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)} = \frac{-2 + 1}{10} = -\frac{1}{10};$$

$$u_1 = (a - \lambda_1)\sin x + b\cos x = \sin x - 2\cos x,$$

$$u_2 = (a - \lambda_2)\sin x + b\cos x = -4\sin x - 2\cos x;$$

$$k_1 = \frac{1}{a - \lambda_1} = 1,$$

$$k_2 = \frac{1}{a - \lambda_2} = -\frac{1}{4}.$$

代入得

$$\int \frac{\sin x + \cos x}{2\sin^2 x - 4\sin x \cos x + 5\cos^2 x} dx$$

$$\begin{aligned}
&= \frac{3}{5} \int \frac{d(\sin x - 2\cos x)}{(\sin x - 2\cos x)^2 + 1} \\
&\quad + \frac{1}{10} \int \frac{d(4\sin x + 2\cos x)}{6 - \frac{1}{4}(4\sin x + 2\cos x)^2} \\
&= \frac{3}{5} \arctan(\sin x - 2\cos x) \\
&\quad + \frac{1}{10\sqrt{6}} \ln \left| \frac{\sqrt{6} + 2\sin x + \cos x}{\sqrt{6} - 2\sin x - \cos x} \right| + C.
\end{aligned}$$

2056. $\int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx.$

解 本题也是 2053 题的特例, 因为

$$\begin{aligned}
&\int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx \\
&= \int \frac{\sin x - 2\cos x}{\sin^2 x + 4\sin x \cos x + \cos^2 x} dx,
\end{aligned}$$

这里,

$$a_1 = 1, b_1 = 2, a = 1, b = 2, c = 1;$$

$$\lambda_1 = 3, \lambda_2 = -1, k_1 = -\frac{1}{2}, k_2 = \frac{1}{2};$$

$$A = \frac{1}{4}, B = -\frac{3}{4};$$

$$u_1 = 2(\cos x - \sin x), u_2 = 2(\cos x + \sin x).$$

代入得

$$\begin{aligned}
&\int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx \\
&= \frac{1}{4} \int \frac{2d(\cos x - \sin x)}{-2(\cos x - \sin x)^2 + 3} \\
&\quad - \frac{3}{4} \int \frac{2d(\cos x + \sin x)}{2(\cos x + \sin x)^2 - 1}
\end{aligned}$$

$$= \frac{3}{4\sqrt{2}} \ln \left| \frac{\sqrt{2}(\sin x + \cos x) + 1}{\sqrt{2}(\sin x + \cos x) - 1} \right| \\ - \frac{1}{4\sqrt{6}} \ln \left| \frac{\sqrt{3} + \sqrt{2}(\sin x - \cos x)}{\sqrt{3} - \sqrt{2}(\sin x - \cos x)} \right| + C.$$

2057. 证明

$$\int \frac{dx}{(a \sin x + b \cos x)^n} = \frac{A \sin x + B \cos x}{(a \sin x + b \cos x)^{n-1}} \\ + C \int \frac{dx}{(a \sin x + b \cos x)^{n-2}},$$

式中 A, B, C 为未定系数.

证 $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \alpha),$

式中 $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}},$

于是,

$$\int \frac{dx}{(a \sin x + b \cos x)^n} = (a^2 + b^2)^{-\frac{n}{2}} \int \frac{dx}{\sin^n(x + \alpha)} \\ = - (a^2 + b^2)^{-\frac{n}{2}} \int \frac{1}{\sin^{n-2}(x + \alpha)} d[\operatorname{ctg}(x + \alpha)] \\ = - (a^2 + b^2)^{-\frac{n}{2}} \frac{\operatorname{ctg}(x + \alpha)}{\sin^{n-2}(x + \alpha)} \\ - \frac{n-2}{(a^2 + b^2)^{\frac{n}{2}}} \int \frac{\operatorname{ctg}(x + \alpha) \cos(x + \alpha)}{\sin^{n-2}(x + \alpha)} dx \\ = \frac{\frac{b}{a^2 + b^2} \sin x - \frac{a}{a^2 + b^2} \cos x}{(a \sin x + b \cos x)^{n-1}} \\ - \frac{n-2}{(a^2 + b^2)^{\frac{n}{2}}} \int \frac{1 - \sin^2(x + \alpha)}{\sin^n(x + \alpha)} dx.$$

设 $I_n = \int \frac{dx}{(a \sin x + b \cos x)^n}$, 则由上式可得

$$I_n = \frac{\frac{b}{a^2 + b^2} \sin x - \frac{a}{a^2 + b^2} \cos x}{(a \sin x + b \cos x)^{n-1}} \\ + (2-n)I_n + \frac{n-2}{a^2 + b^2} I_{n-2}.$$

于是,

$$I_n = \frac{\frac{b}{(n-1)(a^2 + b^2)} \sin x - \frac{a}{(n-1)(a^2 + b^2)} \cos x}{(a \sin x + b \cos x)^{n-1}} \\ + \frac{n-2}{(n-1)(a^2 + b^2)} I_{n-2},$$

即

$$\int \frac{dx}{(a \sin x + b \cos x)^n} = \frac{A \sin x + B \cos x}{(a \sin x + b \cos x)^{n-1}} \\ + C \int \frac{dx}{(a \sin x + b \cos x)^{n-2}},$$

$$\text{式中 } A = \frac{b}{(n-1)(a^2 + b^2)}, B = \frac{a}{(n-1)(a^2 + b^2)},$$

$$C = \frac{n-2}{(n-1)(a^2 + b^2)}.$$

2058. 求 $\int \frac{dx}{(\sin x + 2 \cos x)^3}.$

解 此为 2057 题之特例, 这里

$$a = 1, b = 2, n = 3;$$

$$A = \frac{2}{10}, B = -\frac{1}{10}, C = \frac{1}{10}.$$

代入得

$$\int \frac{dx}{(\sin x + 2 \cos x)^3} = \frac{2 \sin x - \cos x}{10(\sin x + 2 \cos x)^2} \\ + \frac{1}{10} \int \frac{dx}{\sin x + 2 \cos x}$$

$$\begin{aligned}
&= \frac{2\sin x - \cos x}{10(\sin x + 2\cos x)^2} + \frac{1}{10\sqrt{5}} \int \frac{dx}{\sin(x+\alpha)} \\
&= \frac{2\sin x - \cos x}{10(\sin x + 2\cos x)^2} \\
&\quad + \frac{1}{10\sqrt{5}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\alpha}{2} \right) \right| + C.
\end{aligned}$$

其中 $\cos \alpha = \frac{1}{\sqrt{5}}, \sin \alpha = \frac{2}{\sqrt{5}}, \alpha = \operatorname{arctg} 2$.

2059. 若 n 为大于 1 的自然数, 证明

$$\begin{aligned}
&\int \frac{dx}{(a + b\cos x)^n} = \frac{A\sin x}{(a + b\cos x)^{n-1}} \\
&\quad + B \int \frac{dx}{(a + b\cos x)^{n-1}} + C \int \frac{dx}{(a + b\cos x)^{n-2}} \\
&\quad (|a| \neq |b|),
\end{aligned}$$

并求出系数 A, B 和 C .

证 设 $I_n = \int \frac{dx}{(a + b\cos x)^n}$, 先考虑 I_{n-1} .

$$\begin{aligned}
I_{n-1} &= \frac{1}{a} \int \frac{(a + b\cos x) - b\cos x}{(a + b\cos x)^{n-1}} dx \\
&= \frac{1}{a} I_{n-2} - \frac{b}{a} \int \frac{d(\sin x)}{(a + b\cos x)^{n-1}} \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a + b\cos x)^{n-1}} \\
&\quad + \frac{(n-1)b^2}{a} \int \frac{\sin^2 x}{(a + b\cos x)^n} dx \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a + b\cos x)^{n-1}} \\
&\quad + \frac{n-1}{a} \int \frac{(b^2 - a^2) + (a + b\cos x)(a - b\cos x)}{(a + b\cos x)^n} dx
\end{aligned}$$

$$+ B \int \frac{dx}{(a + b \cos x)^{n-1}} + C \int \frac{dx}{(a + b \cos x)^{n-2}}.$$

$$\text{式中 } A = -\frac{b}{(n-1)(a^2 - b^2)}, B = \frac{(2n-3)a}{(n-1)(a^2 - b^2)},$$

$$C = -\frac{n-2}{(n-1)(a^2 - b^2)} (|a| \neq |b|; n > 1 \text{ 且 } a \neq 0).$$

若 $a = 0$, 则 $b \neq 0$, 我们有

$$\begin{aligned} \int \frac{dx}{(a + b \cos x)^n} &= \frac{1}{b^n} \int \frac{dx}{\cos^n x} \\ &= \frac{1}{b^n} \left(\frac{\sin x}{(n-1)\cos^{n-1}x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2}x} \right)^{**}. \end{aligned}$$

*) 利用 2012 题(6) 的结果.

求积分:

$$2060. \int \frac{\sin x dx}{\cos x \sqrt{1 + \sin^2 x}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin x dx}{\cos x \sqrt{1 + \sin^2 x}} &= \int \frac{-d(\cos x)}{\cos x \sqrt{2 - \cos^2 x}} \\ &= - \int \frac{d(\cos x)}{\cos^2 x \sqrt{2 \sec^2 x - 1}} = \int \frac{d(\sec x)}{\sqrt{2 \sec^2 x - 1}} \\ &= \frac{1}{\sqrt{2}} \ln | \sqrt{2} \sec x + \sqrt{2 \sec^2 x - 1} | + C \\ &= \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 + \sin^2 x}}{|\cos x|} + C. \end{aligned}$$

$$2061. \int \frac{\sin^2 x}{\cos^2 x \sqrt{\operatorname{tg} x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin^2 x}{\cos^2 x \sqrt{\operatorname{tg} x}} dx &= \int \frac{\sin^2 x d(\operatorname{tg} x)}{\sqrt{\operatorname{tg} x}} \\ &= 2 \int \sin^2 x d(\sqrt{\operatorname{tg} x}) = 2 \int (1 - \cos^2 x) d(\sqrt{\operatorname{tg} x}) \end{aligned}$$

$$\begin{aligned}
&= 2\sqrt{\operatorname{tg}x} - 2\int \frac{d(\sqrt{\operatorname{tg}x})}{1 + \operatorname{tg}^2x} \\
&= 2\sqrt{\operatorname{tg}x} - \frac{1}{2\sqrt{2}} \ln \frac{\operatorname{tg}x + \sqrt{2\operatorname{tg}x} + 1}{\operatorname{tg}x - \sqrt{2\operatorname{tg}x} + 1} \\
&\quad + \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{\sqrt{2\operatorname{tg}x}}{\operatorname{tg}x - 1} + C \quad (\operatorname{tg}x > 0).
\end{aligned}$$

*) 利用 1884 题的结果.

2062. $\int \frac{\sin x dx}{\sqrt{2 + \sin 2x}}.$

解 由于

$$\begin{aligned}
2 + \sin 2x &= 1 + (\sin x + \cos x)^2 \\
&= 3 - (\sin x - \cos x)^2.
\end{aligned}$$

于是,

$$\begin{aligned}
\int \frac{\sin x dx}{\sqrt{2 + \sin 2x}} &= \int \frac{\cos x - (\cos x - \sin x)}{\sqrt{1 + (\sin x + \cos x)^2}} dx \\
&= \int \frac{\cos x dx}{\sqrt{3 - (\sin x - \cos x)^2}} dx \\
&\quad - \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}) \\
&= - \int \frac{\sin x dx}{\sqrt{2 + \sin 2x}} + \int \frac{d(\sin x - \cos x)}{\sqrt{3 - (\sin x - \cos x)^2}} \\
&\quad - \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}),
\end{aligned}$$

因而,

$$\begin{aligned}
\int \frac{\sin x dx}{\sqrt{2 + \sin 2x}} &= \frac{1}{2} \int \frac{d(\sin x - \cos x)}{\sqrt{3 - (\sin x - \cos x)^2}} \\
&\quad - \frac{1}{2} \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}) \\
&= \frac{1}{2} \arcsin \left(\frac{\sin x - \cos x}{\sqrt{3}} \right)
\end{aligned}$$

$$= -\frac{1}{2} \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}) + C.$$

2063. $\int \frac{dx}{(1 + \epsilon \cos x)^2} (0 < \epsilon < 1).$

解 此为 2059 题之特例, 这里

$$a = 1, b = \epsilon, n = 2,$$

$$A = -\frac{\epsilon}{1 - \epsilon^2}, B = \frac{1}{1 - \epsilon^2}, C = 0.$$

代入得

$$\begin{aligned} \int \frac{dx}{(1 + \epsilon \cos x)^2} &= -\frac{\epsilon \sin x}{(1 - \epsilon^2)(1 + \epsilon \cos x)} \\ &+ \frac{1}{1 - \epsilon^2} \int \frac{1}{1 + \epsilon \cos x} \\ &= -\frac{\epsilon \sin x}{(1 - \epsilon^2)(1 + \epsilon \cos x)} \\ &+ \frac{2}{(1 - \epsilon^2)^{\frac{3}{2}}} \operatorname{arctg} \left(\sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \operatorname{tg} \frac{x}{2} \right)^{*)} + C. \end{aligned}$$

*) 利用 2028 题(a) 的结果.

2064⁺. $\int \frac{\cos^{n-1} \frac{x+a}{2}}{\sin^{n+1} \frac{x-a}{2}} dx$

解 设 $t = \frac{\cos \frac{x+a}{2}}{\sin \frac{x-a}{2}}$, 则

$$dt = \frac{-\frac{1}{2} \cos a}{\sin^2 \frac{x-a}{2}} dx, \quad \frac{dx}{\sin^2 \frac{x-a}{2}} = -\frac{2}{\cos a} dt.$$

于是,

$$\begin{aligned}
& \int \frac{\cos^{n-1} \frac{x+a}{2}}{\sin^{n+1} \frac{x-a}{2}} dx = -\frac{2}{\cos a} \int t^{n-1} dt \\
& = -\frac{2}{n \cos a} t^n + C \\
& = -\frac{2}{n \cos a} \left[\frac{\cos \frac{x+a}{2}}{\sin \frac{x-a}{2}} \right]^n + C (\cos a \neq 0).
\end{aligned}$$

2065. 推出积分

$$I_n = \int \left[\frac{\sin \frac{x-a}{2}}{\sin \frac{x+a}{2}} \right]^n dx$$

的递推公式(n 为自然数).

证 方法一:

$$\text{设 } t = \frac{\sin \frac{x-a}{2}}{\sin \frac{x+a}{2}}, \text{ 则}$$

$$x = 2 \operatorname{arctg} \left(\frac{1+t}{1-t} \cdot \operatorname{tg} \frac{a}{2} \right),$$

$$dx = \frac{4 \operatorname{tg} \frac{a}{2}}{t^2 \sec^2 \frac{a}{2} + 2t \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right) + \sec^2 \frac{a}{2}} dt.$$

由于

$$\frac{4t^n \operatorname{tg} \frac{a}{2}}{t^2 \sec^2 \frac{a}{2} + 2t \cdot \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right) + \sec^2 \frac{a}{2}}$$

$$\begin{aligned}
&= \frac{4\operatorname{tg} \frac{a}{2}}{\sec^2 \frac{a}{2}} t^{n-2} + \frac{-4\operatorname{tg} \frac{a}{2}}{t^2 \sec^2 \frac{a}{2} + 2t \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right) + \sec^2 \frac{a}{2}} \\
&\quad \cdot \frac{2 \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right)}{\sec^2 \frac{a}{2}} t^{n-1} \\
&\quad + \frac{-4\operatorname{tg} \frac{a}{2}}{t^2 \sec^2 \frac{a}{2} + 2t \cdot \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right) + \sec^2 \frac{a}{2}} \\
&\quad \cdot t^{n-2} \quad (n > 2),
\end{aligned}$$

两端对 t 积分, 即得递推公式

$$I_n = \frac{2\sin a}{n-1} t^{n-1} + 2I_{n-1} \cos a - I_{n-2}.$$

方法二:

设 $y = \frac{x+a}{2}$, 则 $\frac{x-a}{2} = y-a$, 从而

$$\begin{aligned}
I_n &= 2 \int \left(\frac{\sin(y-a)}{\sin y} \right)^n dy \\
&= 2 \int \frac{\sin(y-a)}{\sin y} \left(\frac{\sin(y-a)}{\sin y} \right)^{n-1} dy \\
&= 2 \int \frac{\sin y \cos a - \cos y \sin a}{\sin y} \left(\frac{\sin(y-a)}{\sin y} \right)^{n-1} dy \\
&= \cos a I_{n-1} - 2 \sin a \int \frac{\cos y}{\sin y} \left(\frac{\sin(y-a)}{\sin y} \right)^{n-1} dy.
\end{aligned}$$

再设

$$\frac{\sin(y-a)}{\sin y} = t = \frac{\sin \left(\frac{x-a}{2} \right)}{\sin \frac{x+a}{2}}, J_n = 2 \int \frac{\cos y}{\sin y} t^n dy,$$

则

$$I_n = \cos a I_{n-1} - \sin a J_{n-1}, J_{n-1} = \frac{\cos a I_{n-1} - I_n}{\sin a}. \quad (1)$$

又

$$\begin{aligned} J_n &= 2 \int \frac{\cos y}{\sin y} \left(\frac{\sin(y-a)}{\sin y} \right)^n dy \\ &= -\frac{2}{n} \int \sin^n(y-a) d\left(\frac{1}{\sin^n y} \right) \\ &= -\frac{2}{n} \left[\frac{\sin^n(y-a)}{\sin y} \right] \\ &\quad + 2 \int \frac{\sin^{n-1}(y-a)}{\sin^n y} \cos(y-a) dy \\ &= -\frac{2}{n} t^n + 2 \int t^{n-1} \frac{\cos y \cos a + \sin y \sin a}{\sin y} dy \\ &= -\frac{2}{n} t^n + \cos a J_{n-1} + \sin a I_{n-1}. \end{aligned} \quad (2)$$

由(1)式和(2)式解得

$$\begin{aligned} I_n &= I_{n-1} \cos a - \sin a J_{n-1} \\ &= I_{n-1} \cos a - \sin a \left(-\frac{2}{n-1} t^{n-1} \right. \\ &\quad \left. + \cos a J_{n-2} + I_{n-2} \sin a \right) \\ &= I_{n-1} \cos a + \frac{2 \sin a}{n-1} t^{n-1} \\ &\quad - \sin a \cos a \left(\frac{I_{n-2} \cos a - I_{n-1}}{\sin a} \right) - \sin^2 a I_{n-2} \\ &= 2 I_{n-1} \cos a - I_{n-2} + \frac{2 \sin a}{n-1} t^{n-1}. \end{aligned}$$

§ 5. 各种超越函数的积分法

2066. 证明若 $P(x)$ 为 n 次多项式, 则

$$\int P(x)e^{ax}dx = e^{ax}\left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} + \dots + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}}\right] + C.$$

证

$$\begin{aligned} \int P(x)e^{ax}dx &= \frac{1}{a} \int P(x)d(e^{ax}) \\ &= \frac{1}{a} P(x)e^{ax} - \frac{1}{a} \int e^{ax} P'(x)dx \\ &= \frac{1}{a} P(x)e^{ax} - \frac{1}{a^2} \int P'(x)d(e^{ax}) \\ &= \frac{1}{a} P(x)e^{ax} - \frac{1}{a^2} P'(x)e^{ax} + \frac{1}{a^2} \int e^{ax} P''(x)dx \\ &= \dots\dots \\ &= e^{ax}\left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} + \dots + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}}\right] \\ &\quad + C. \end{aligned}$$

因为 $P(x)$ 为 n 次多项式, 所以 $P^{(n+1)}(x) \equiv 0$, 从而上述等式括号中的导数到 $P^{(n)}(x)$ 为止.

2067. 证明若 $P(x)$ 为 n 次多项式, 则

$$\begin{aligned} &\int P(x)\cos ax dx \\ &= \frac{\sin ax}{a} \left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots \right] \\ &\quad + \frac{\cos ax}{a^2} \left[P'(x) + \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots \right] + C \end{aligned}$$

及

$$\begin{aligned} &\int P(x)\sin ax dx \\ &= -\frac{\cos ax}{a} \left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots \right] \\ &\quad + \frac{\sin ax}{a^2} \left[P'(x) + \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots \right] + C \end{aligned}$$

$$+ \frac{\sin ax}{a^2} \left(P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots \right) + C.$$

证 $\int P(x) \cos ax dx = \frac{1}{a} \int P(x) d(\sin ax)$

$$= \frac{1}{a} P(x) \sin ax - \frac{1}{a} \int P'(x) \sin ax dx$$

$$= \frac{1}{a} P(x) \sin ax + \frac{1}{a^2} \int P'(x) d(\cos ax)$$

$$= \frac{1}{a} P(x) \sin ax + \frac{1}{a^2} P'(x) \cos ax$$

$$- \frac{1}{a^2} \int P''(x) \cos ax dx$$

$$= \frac{1}{a} P(x) \sin ax + \frac{1}{a^2} P'(x) \cos ax - \frac{1}{a^3} P''(x) \sin ax$$

$$- \frac{1}{a^4} P'''(x) \cos ax + \frac{1}{a^4} \int P^{(4)}(x) \cos ax dx$$

$$= \dots\dots$$

$$= \frac{\sin ax}{a} \left(P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots \right)$$

$$+ \frac{\cos ax}{a^2} \left(P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots \right)$$

$$+ C.$$

$$\int P(x) \sin ax dx = - \frac{1}{a} \int P(x) d(\cos ax)$$

$$= - \frac{1}{a} P(x) \cos ax + \frac{1}{a} \int P'(x) \cos ax dx$$

$$= - \frac{1}{a} P(x) \cos ax + \frac{1}{a^2} \int P'(x) d(\sin ax)$$

$$= - \frac{1}{a} P(x) \cos ax + \frac{1}{a^2} P'(x) \sin ax$$

$$- \frac{1}{a^2} \int P''(x) \sin ax dx$$

$$\begin{aligned}
&= -\frac{1}{a}P(x)\cos ax + \frac{1}{a^2}P'(x)\sin ax + \frac{1}{a^3}P''(x)\cos ax \\
&\quad - \frac{1}{a^4}P'''(x)\sin ax + \frac{1}{a^4}\int P^{(4)}(x)\sin ax dx \\
&= \dots\dots \\
&= -\frac{\cos ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots\right] \\
&\quad + \frac{\sin ax}{a^2}\left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots\right] \\
&\quad + C.
\end{aligned}$$

上述导数项是有限的, 其次数 $\leq n$.

求积分:

2068. $\int x^3 e^{3x} dx.$

解 $\int x^3 e^{3x} dx = e^{3x} \left(\frac{x^3}{3} - \frac{3x^2}{9} + \frac{6x}{27} - \frac{6}{81} \right)^{**} + C$
 $= e^{3x} \left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{2}{27} \right) + C.$

*) 利用 2066 题的结果.

2069. $\int (x^2 - 2x + 2)e^{-x} dx.$

解 $\int (x^2 - 2x + 2)e^{-x} dx = e^{-x} \left(\frac{x^2 - 2x + 2}{-1} \right.$
 $\left. - \frac{2x - 2}{1} + \frac{2}{-1} \right)^{**} + C$
 $= -e^{-x}(x^2 + 2) + C.$

*) 利用 2066 题的结果.

2070. $\int x^5 \sin 5x dx.$

解 $\int x^5 \sin 5x dx = -\frac{\cos 5x}{5} \left(x^5 - \frac{20x^3}{25} + \frac{120x}{625} \right)$

*) 利用 2066 题的结果.

$$2074. \int e^{ax} \cos^2 bx dx.$$

$$\begin{aligned} \text{解} \quad \int e^{ax} \cos^2 bx dx &= \frac{1}{2} \int e^{ax} (1 + \cos 2bx) dx \\ &= \frac{1}{2a} e^{ax} + \frac{1}{2} e^{ax} \cdot \frac{a \cos 2bx + 2b \sin 2bx^{**}}{a^2 + 4b^2} + C. \end{aligned}$$

*) 利用 1828 题的结果.

$$2075. \int e^{ax} \sin^3 bx dx.$$

$$\begin{aligned} \text{解} \quad \int e^{ax} \sin^3 bx dx &= \int e^{ax} \sin bx \cdot \frac{1 - \cos 2bx}{2} dx \\ &= \int e^{ax} \left(\frac{3}{4} \sin bx - \frac{1}{4} \sin 3bx \right) dx \\ &= \frac{3}{4} e^{ax} \cdot \frac{a \sin bx - b \cos bx^{**}}{a^2 + b^2} \\ &\quad - \frac{1}{4} e^{ax} \cdot \frac{a \sin 3bx - 3b \cos 3bx^{**}}{a^2 + 9b^2} + C. \end{aligned}$$

*) 利用 1829 题的结果.

$$2076. \int x e^x \sin x dx.$$

$$\begin{aligned} \text{解} \quad \int x e^x \sin x dx &= \int x \sin x d(e^x) \\ &= x e^x \sin x - \int e^x (\sin x + x \cos x) dx \\ &= x e^x \sin x - \int (\sin x + x \cos x) d(e^x) \\ &= e^x (x \sin x - \sin x - x \cos x) \\ &\quad + \int e^x (2 \cos x - x \sin x) dx \\ &= e^x (x \sin x - \sin x - x \cos x) \end{aligned}$$

$$+ 2 \int e^x \cos x dx - \int x e^x \sin x dx,$$

于是,

$$\begin{aligned} \int x e^x \sin x dx &= \frac{e^x}{2} (x \sin x - \sin x \\ &\quad - x \cos x) + \int e^x \cos x dx \\ &= \frac{e^x}{2} (x \sin x - \sin x - x \cos x) + \frac{e^x}{2} (\sin x \\ &\quad + \cos x) *) + C \\ &= \frac{e^x}{2} [x(\sin x - \cos x) + \cos x] + C. \end{aligned}$$

*) 利用 1828 题的结果.

2077. $\int x^2 e^x \cos x dx.$

$$\begin{aligned} \text{解} \quad \int x^2 e^x \cos x dx &= \int x^2 \cos x d(e^x) \\ &= x^2 e^x \cos x - \int e^x (2x \cos x - x^2 \sin x) dx \\ &= x^2 e^x \cos x - \int (2x \cos x - x^2 \sin x) d(e^x) \\ &= x^2 e^x \cos x - e^x (2x \cos x - x^2 \sin x) \\ &\quad + \int e^x (2 \cos x - 4x \sin x - x^2 \cos x) dx \\ &= e^x [x^2 (\sin x + \cos x) - 2x \cos x] + 2 \int e^x \cos x dx \\ &\quad - 4 \int x e^x \sin x dx - \int x^2 e^x \cos x dx, \end{aligned}$$

于是,

$$\int x^2 e^x \cos x dx = \frac{e^x}{2} [x^2 (\sin x + \cos x) - 2x \cos x]$$

$$\begin{aligned}
& + \int e^x \cos x dx - 2 \int x e^x \sin x dx \\
& = \frac{e^x}{2} [x^2 (\sin x + \cos x) - 2x \cos x] + \frac{e^x}{2} (\sin x \\
& \quad + \cos x)^{**} - 2 \cdot \frac{e^x}{2} [x (\sin x - \cos x) + \cos x]^{***} \\
& \quad + C.
\end{aligned}$$

$$\begin{aligned}
& = \frac{e^x}{2} [x^2 (\sin x + \cos x) - 2x \sin x \\
& \quad + (\sin x - \cos x)] + C.
\end{aligned}$$

*) 利用 1828 题的结果.

* *) 利用 2076 题的结果.

2078. $\int x e^x \sin^2 x dx.$

$$\begin{aligned}
\text{解} \quad \int x e^x \sin^2 x dx &= \frac{1}{2} \int x e^x (1 - \cos 2x) dx \\
&= \frac{1}{2} \int x e^x dx - \frac{1}{2} \int x e^x \cos 2x dx \\
&= \frac{1}{2} e^x (x - 1) - \frac{1}{2} \int x \cos 2x d(e^x) \\
&= \frac{1}{2} e^x (x - 1) - \frac{1}{2} x e^x \cos 2x \\
& \quad + \frac{1}{2} \int e^x (\cos 2x - 2x \sin 2x) dx \\
&= \frac{1}{2} e^x (x - 1) - \frac{1}{2} x e^x \cos 2x \\
& \quad + \frac{e^x}{2} \cdot \frac{\cos 2x + 2 \sin 2x^{**}}{5} - \int x e^x \sin 2x dx,
\end{aligned}$$

而

$$\int x e^x \sin 2x dx = \int x \sin 2x d(e^x)$$

$$\begin{aligned}
&= xe^x \sin 2x - \int e^x (\sin 2x + 2x \cos 2x) dx \\
&= xe^x \sin 2x - \frac{e^x}{5} (\sin 2x - 2 \cos 2x)^{**}) \\
&\quad - 2 \int xe^x (1 - 2 \sin^2 x) dx \\
&= xe^x \sin 2x - \frac{e^x}{5} (\sin 2x - 2 \cos 2x) \\
&\quad - 2(x-1)e^x + 4 \int xe^x \sin^2 x dx.
\end{aligned}$$

代入得

$$\begin{aligned}
\int xe^x \sin^2 x dx &= e^x \left[\frac{x-1}{2} - \frac{x}{10} (2 \sin 2x + \cos 2x) \right. \\
&\quad \left. + \frac{1}{50} (4 \sin 2x - 3 \cos 2x) \right] + C.
\end{aligned}$$

*) 利用 1828 题的结果.

**) 利用 1829 题的结果.

2079. $\int (x - \sin x)^3 dx.$

$$\begin{aligned}
\text{解 } \int (x - \sin x)^3 dx &= \int (x^3 - 3x^2 \sin x + 3x \sin^2 x - \sin^3 x) dx \\
&= \frac{x^4}{4} + 3 \int x^2 d(\cos x) + \frac{3}{2} \int x(1 - \cos 2x) dx \\
&\quad + \int (1 - \cos^2 x) d(\cos x) \\
&= \frac{x^4}{4} + 3x^2 \cos x - 6 \int x \cos x dx + \frac{3}{4} x^2 \\
&\quad - \frac{3}{4} \int x d(\sin 2x) + \cos x - \frac{1}{3} \cos^3 x \\
&= \frac{x^4}{4} + \frac{3x^2}{4} + 3x^2 \cos x - 6 \int x d(\sin x) - \frac{3}{4} x \sin 2x
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{4} \int \sin 2x dx + \cos x - \frac{1}{3} \cos^3 x \\
& = \frac{x^4}{4} + \frac{3x^2}{4} + 3x^2 \cos x - 6x \sin x - 6 \cos x \\
& \quad - \frac{3}{4} x \sin 2x + \cos x - \frac{3}{8} \cos 2x - \frac{1}{3} \cos^3 x + C \\
& = \frac{x^4}{4} + \frac{3x^2}{4} + 3x^2 \cos x - x \left(6 \sin x + \frac{3}{4} \sin 2x \right) \\
& \quad - \left(5 \cos x + \frac{3}{8} \cos 2x \right) - \frac{1}{3} \cos^3 x + C.
\end{aligned}$$

2080. $\int \cos^2 \sqrt{x} dx.$

解 设 $\sqrt{x} = t$, 则 $x = t^2, dx = 2t dt$. 于是

$$\begin{aligned}
\int \cos^2 \sqrt{x} dx &= 2 \int t \cos^2 t dt = \int t(1 + \cos 2t) dt \\
&= \frac{t^2}{2} + \frac{1}{2} \int t d(\sin 2t) \\
&= \frac{t^2}{2} + \frac{1}{2} t \sin 2t - \frac{1}{2} \int \sin 2t dt \\
&= \frac{t^2}{2} + \frac{1}{2} t \sin 2t + \frac{1}{4} \cos 2t + C \\
&= \frac{x}{2} + \frac{1}{2} \sqrt{x} \sin(2\sqrt{x}) + \frac{1}{4} \cos(2\sqrt{x}) + C.
\end{aligned}$$

2081. 证明若 R 为有理函数及 a_1, a_2, \dots, a_n 为可公度的数, 则积分

$$\int R(e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}) dx$$

是初等函数.

证 按题意 a_1, a_2, \dots, a_n 为可公度的数, 于是存在一个实数 α , 使得

$$a_1 = k_1 \alpha, a_2 = k_2 \alpha, \dots, a_n = k_n \alpha (\alpha \neq 0),$$

其中, k_1, k_2, \dots, k_n 为整数.

设 $e^{ax} = t$, 则 $x = \frac{1}{a} \ln t, dx = \frac{1}{at} dt$.

于是,

$$\begin{aligned} & \int R(e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}) dx \\ &= \frac{1}{a} \int R(t^{k_1}, t^{k_2}, \dots, t^{k_n}) \frac{dt}{t} = \int R^*(t) dt, \end{aligned}$$

其中 $R^*(t)$ 是 t 的有理函数. 因此, 积分

$$\int R(e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}) dx$$

为初等函数.

求下列积分:

2082. $\int \frac{dx}{(1+e^x)^2}.$

解
$$\begin{aligned} \int \frac{dx}{(1+e^x)^2} &= \int \frac{(1+e^x) - e^x}{(1+e^x)^2} dx \\ &= \int \frac{dx}{1+e^x} - \int \frac{e^x dx}{(1+e^x)^2} \\ &= \int \left(1 - \frac{e^x}{1+e^x} \right) dx - \int \frac{d(1+e^x)}{(1+e^x)^2} \\ &= x - \ln(1+e^x) + \frac{1}{1+e^x} + C. \end{aligned}$$

2083. $\int \frac{e^{2x} dx}{1+e^x}.$

解
$$\begin{aligned} \int \frac{e^{2x} dx}{1+e^x} &= \int \frac{(e^{2x} - 1) + 1}{1+e^x} dx \\ &= \int (e^x - 1) dx + \int \frac{1}{1+e^x} dx \\ &= e^x - x + \int \left(1 - \frac{e^x}{1+e^x} \right) dx \end{aligned}$$

$$= e^x - \ln(1 + e^x) + C.$$

$$2084. \int \frac{dx}{e^{2x} + e^x - 2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{e^{2x} + e^x - 2} &= \int \frac{dx}{(e^x + 2)(e^x - 1)} \\ &= \frac{1}{3} \int \frac{1}{e^x - 1} dx - \frac{1}{3} \int \frac{1}{e^x + 2} dx \\ &= -\frac{1}{3} \int \left(1 - \frac{e^x}{e^x - 1} \right) dx - \frac{1}{6} \int \left(1 - \frac{e^x}{e^x + 2} \right) dx \\ &= -\frac{x}{3} + \frac{1}{3} \ln|e^x - 1| - \frac{x}{6} + \frac{1}{6} \ln(e^x + 2) + C \\ &= -\frac{x}{2} + \frac{1}{3} \ln|e^x - 1| + \frac{1}{6} \ln(e^x + 2) + C. \end{aligned}$$

$$2085. \int \frac{dx}{1 + e^{\frac{x}{2}} + e^{\frac{x}{3}} + e^{\frac{x}{6}}}.$$

$$\text{解} \quad \text{设 } e^{\frac{x}{6}} = t, \text{ 则 } x = 6 \ln t, dx = \frac{6}{t} dt.$$

代入得

$$\begin{aligned} \int \frac{dx}{1 + e^{\frac{x}{2}} + e^{\frac{x}{3}} + e^{\frac{x}{6}}} &= 6 \int \frac{dt}{t(1 + t^3 + t^2 + t)} \\ &= 6 \int \frac{dt}{t(t+1)(t^2+1)} \\ &= 6 \int \left(\frac{1}{t} - \frac{1}{2(t+1)} - \frac{t+1}{2(t^2+1)} \right) dt \\ &= 6 \ln t - 3 \ln(t+1) - \frac{3}{2} \ln(1+t^2) \\ &\quad - 3 \arctan t + C \\ &= x - 3 \ln \left[(1 + e^{\frac{x}{6}}) \sqrt{1 + e^{\frac{x}{3}}} \right] - 3 \arctan e^{\frac{x}{6}} + C. \end{aligned}$$

$$2086. \int \frac{1 + e^{\frac{x}{2}}}{(1 + e^{\frac{x}{4}})^2} dx.$$

解 设 $e^{\frac{x}{4}} = t$, 则 $x = 4\ln t, dx = \frac{4}{t}dt$.

代入得

$$\begin{aligned}\int \frac{1 + e^{\frac{x}{2}}}{(1 + e^{\frac{x}{4}})^2} dx &= 4 \int \frac{1 + t^2}{t(1 + t)^2} dt \\ &= 4 \int \left[\frac{1}{t} - \frac{2}{(1 + t)^2} \right] dt \\ &= 4\ln t + \frac{8}{1 + t} + C = x + \frac{8}{1 + e^{\frac{x}{4}}} + C.\end{aligned}$$

2087. $\int \frac{dx}{\sqrt{e^x - 1}}.$

解
$$\begin{aligned}\int \frac{dx}{\sqrt{e^x - 1}} &= \int \frac{dx}{e^{\frac{x}{2}} \sqrt{1 - (e^{-\frac{x}{2}})^2}} \\ &= -2 \int \frac{d(e^{-\frac{x}{2}})}{\sqrt{1 - (e^{-\frac{x}{2}})^2}} \\ &= -2 \arcsin(e^{-\frac{x}{2}}) + C.\end{aligned}$$

2088. $\int \sqrt{\frac{e^x - 1}{e^x + 1}} dx.$

解
$$\begin{aligned}\int \sqrt{\frac{e^x - 1}{e^x + 1}} dx &= \int \frac{e^x - 1}{\sqrt{e^{2x} - 1}} dx \\ &= \int \frac{e^x dx}{\sqrt{e^{2x} - 1}} - \int \frac{dx}{\sqrt{e^{2x} - 1}} \\ &= \int \frac{d(e^x)}{\sqrt{(e^x)^2 - 1}} + \int \frac{d(e^{-x})}{\sqrt{1 - (e^{-x})^2}} \\ &= \ln(e^x + \sqrt{e^{2x} - 1}) + \arcsin(e^{-x}) + C.\end{aligned}$$

2089. $\int \sqrt{e^{2x} + 4e^x - 1} dx.$

$$\begin{aligned}
\text{解} \quad & \int \sqrt{e^{2x} + 4e^x - 1} dx = \int \frac{e^{2x} + 4e^x - 1}{\sqrt{e^{2x} + 4e^x - 1}} dx \\
&= \int \frac{2e^{2x} + 4e^x}{2\sqrt{e^{2x} + 4e^x - 1}} dx \\
&\quad + 2 \int \frac{e^x dx}{\sqrt{e^{2x} + 4e^x - 1}} - \int \frac{dx}{\sqrt{e^{2x} + 4e^x - 1}} \\
&= \int \frac{d(e^{2x} + 4e^x - 1)}{2\sqrt{e^{2x} + 4e^x - 1}} + 2 \int \frac{d(e^x + 2)}{\sqrt{(e^x + 2)^2 - 5}} \\
&\quad + \int \frac{d(e^{-x} - 2)}{\sqrt{5 - (e^{-x} - 2)^2}} \\
&= \sqrt{e^{2x} + 4e^x - 1} + 2 \ln(e^x + 2 \\
&\quad + \sqrt{e^{2x} + 4e^x - 1}) - \arcsin \frac{2e^x - 1}{\sqrt{5}e^x} + C.
\end{aligned}$$

$$2090. \int \frac{dx}{\sqrt{1+e^x} + \sqrt{1-e^x}}.$$

$$\begin{aligned}
\text{解} \quad & \int \frac{dx}{\sqrt{1+e^x} + \sqrt{1-e^x}} \\
&= \frac{1}{2} \int e^{-x} (\sqrt{1+e^x} - \sqrt{1-e^x}) dx \\
&= -\frac{1}{2} \int (\sqrt{1+e^x} - \sqrt{1-e^x}) d(e^{-x}) \\
&= -\frac{e^{-x}}{2} (\sqrt{1+e^x} - \sqrt{1-e^x}) \\
&\quad + \frac{1}{4} \int \left(\frac{1}{\sqrt{1+e^x}} + \frac{1}{\sqrt{1-e^x}} \right) dx \\
&= -\frac{1}{2} e^{-x} (\sqrt{1+e^x} - \sqrt{1-e^x}) + \frac{1}{4} I_1 + \frac{1}{4} I_2,
\end{aligned}$$

对于 $I_1 = \int \frac{dx}{\sqrt{1+e^x}}$, 设 $\sqrt{1+e^x} = t$, 则

$$\int \frac{e^{ax}}{x} dx = \operatorname{li}(e^{ax}) + C, \text{ 式中 } \operatorname{li} x = \int \frac{dx}{\ln x}$$

来表示.

证 因为 R 的分母仅有实根, 所以仅包含形如 $(x - a_i)_{k_i}$ 的因子 ($i = 1, 2, \dots, l$). 分解 $R(x)$ 为部分分式得

$$R(x) = P(x) + \sum_{i=1}^l \sum_{j=1}^{k_i} \frac{A_{ij}}{(x - a_i)^j},$$

其中 $P(x)$ 为 x 的多项式, A_{ij} 是常系数.

从而积分

$$\begin{aligned} & \int R(x) e^{ax} dx \\ &= \int P(x) e^{ax} dx + \sum_{i=1}^l \sum_{j=1}^{k_i} A_{ij} \int \frac{e^{ax}}{(x - a_i)^j} dx. \end{aligned}$$

上式右端第一个积分显然是初等函数, 而积分

$$\int \frac{e^{ax}}{(x - a_i)^j} dx$$

可用初等函数和超越函数来表示. 事实上, 设 $x - a_i = t$, 则

$$\begin{aligned} & \int \frac{e^{ax}}{(x - a_i)^j} dx = \int \frac{e^{a(a_i+t)}}{t^j} dt \\ &= \frac{e^{aa_i}}{1-j} \int e^{at} d\left(\frac{1}{t^{j-1}}\right) \\ &= \frac{e^{aa_i}}{1-j} e^t \cdot \frac{1}{t^{j-1}} - \frac{ae^{aa_i}}{1-j} \int \frac{e^{at}}{t^{j-1}} dt. \end{aligned}$$

这样, 被积函数中分母的次数便降低一次, 再继续运用分部积分法 ($j-2$) 次, 即可得

$$\int \frac{e^{ax}}{(x - a_i)^j} dx = g_{ij}(x) + B_{ij} \operatorname{li}(e^{a(x-a_i)}),$$

其中 $g_{ij}(x)$ 为 x 的初等函数, B_{ij} 为常数.

因而积分

$$\int R(x)e^{ax}dx = \int P(x)e^{ax}dx + \sum_{i=1}^l \sum_{j=1}^{k_i} A_{ij}g_{ij}(x) + \sum_{i=1}^l \sum_{j=1}^{k_i} A_{ij}B_{ij}li(e^{a(x-a_i)})$$

是初等函数与超越函数之和.

2092. 在甚么情形下, 积分

$$\int P\left(\frac{1}{x}\right)e^x dx$$

(式中 $P\left(\frac{1}{x}\right) = a_0 + \frac{a_1}{x} + \dots + \frac{a_n}{x^n}$ 及 a_0, a_1, \dots, a_n 为常数) 为初等函数?

$$\begin{aligned} \text{解} \quad \int \frac{a_k}{x^k} e^x dx &= -\frac{a_k}{k-1} \cdot \frac{e^x}{x^{k-1}} + \frac{a_k}{k-1} \int \frac{e^x}{x^{k-1}} dx \\ &= \dots = -\frac{a_k}{k-1} \cdot \frac{e^x}{x^{k-1}} - \frac{a_k}{(k-1)(k-2)} \cdot \frac{e^x}{x^{k-2}} - \dots \\ &\quad - \frac{a_k}{(k-1)!} \cdot \frac{e^x}{x} + \frac{a_k}{(k-1)!} \int \frac{e^x}{x} dx, \end{aligned}$$

于是,

$$\begin{aligned} \int P\left(\frac{1}{x}\right)e^x dx &= \int \left(\sum_{k=0}^n \frac{a_k}{x^k} \right) e^x dx = \sum_{k=0}^n \int \frac{a_k}{x^k} dx \\ &= - \sum_{k=2}^n \sum_{j=1}^{k-1} \frac{a_k}{(k-1)(k-2)\dots(k-j)} \cdot \frac{e^x}{x^{k-j}} \\ &\quad + \sum_{k=1}^n \frac{a_k}{(k-1)!} \int \frac{e^x}{x} dx + a_0 e^x. \end{aligned}$$

因而, 若

$$\sum_{k=1}^n \frac{a_k}{(k-1)!} = 0,$$

即

$$a_1 + \frac{a_2}{1!} + \frac{a_3}{2!} + \cdots + \frac{a_n}{(n-1)!} = 0,$$

则积分 $\int P\left(\frac{1}{x}\right) e^x dx$ 是初等函数.

求积分:

$$2093. \int \left(1 - \frac{2}{x}\right)^2 e^x dx$$

$$\begin{aligned} \text{解} \quad \int \left(1 - \frac{2}{x}\right)^2 e^x dx &= \int \left(1 - \frac{4}{x} + \frac{4}{x^2}\right) e^x dx \\ &= e^x - 4li(e^x) - 4 \int e^x d\left(\frac{1}{x}\right) \\ &= e^x - 4li(e^x) - \frac{4}{x} e^x + 4 \int \frac{e^x}{x} dx \\ &= e^x \left(1 - \frac{4}{x}\right) + C. \end{aligned}$$

$$2094. \int \left(1 - \frac{1}{x}\right) e^{-x} dx.$$

$$\text{解} \quad \int \left(1 - \frac{1}{x}\right) e^{-x} dx = -e^{-x} - li(e^{-x}) + C.$$

$$2095. \int \frac{e^{2x}}{x^2 - 3x + 2} dx$$

$$\begin{aligned} \text{解} \quad \int \frac{e^{2x}}{x^2 - 3x + 2} dx &= \int \frac{e^{2x}}{(x-2)(x-1)} dx \\ &= \int \frac{e^{2x}}{x-2} dx - \int \frac{e^{2x}}{x-1} dx \\ &= e^4 \int \frac{e^{2(x-2)}}{x-2} d(x-2) - e^2 \int \frac{e^{2(x-1)}}{x-1} d(x-1) \\ &= e^4 li(e^{2x-4}) - e^2 li(e^{2x-2}) + C. \end{aligned}$$

$$2096. \int \frac{x e^x}{(x+1)^2} dx.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x e^x}{(x+1)^2} dx &= - \int x e^x d\left(\frac{1}{x+1}\right) \\
 &= - x e^x \frac{1}{x+1} + \int e^x dx = - \frac{x e^x}{x+1} + e^x + C \\
 &= \frac{e^x}{x+1} + C.
 \end{aligned}$$

$$2097. \int \frac{x^4 e^{2x}}{(x-2)^2} dx.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x^4 e^{2x}}{(x-2)^2} dx &= \int (x^2 + 4x + 12) e^{2x} dx \\
 &\quad + 32 \int \frac{e^{2x} dx}{x-2} + 16 \int \frac{e^{2x} dx}{(x-2)^2} \\
 &= e^{2x} \left(\frac{x^2}{2} + \frac{3x}{2} + \frac{21}{4} \right) + 32 e^4 \operatorname{li}(e^{2x-4}) \\
 &\quad - 16 \int e^{2x} d\left(\frac{1}{x-2}\right) \\
 &= \frac{e^{2x}}{2} \left(x^2 + 3x + \frac{21}{2} \right) + 32 e^4 \operatorname{li}(e^{2x-4}) \\
 &\quad - \frac{16 e^{2x}}{x-2} + 32 \int \frac{e^{2x} dx}{x-2} \\
 &= \frac{e^{2x}}{2} \left(x^2 + 3x + \frac{21}{2} - \frac{32}{x-2} \right) + 64 e^4 \operatorname{li}(e^{2x-4}) \\
 &\quad + C.
 \end{aligned}$$

*) 利用 2066 题的结果.

求含有 $\ln f(x)$, $\operatorname{arctg} f(x)$, $\operatorname{arc} \sin f(x)$, $\operatorname{arccos} f(x)$ 等函数的积分, 其中 $f(x)$ 为代数函数:

$$2098. \int \ln^n x dx \quad (n \text{ 为自然数}).$$

$$\begin{aligned}
 \text{解} \quad \int \ln^n x dx &= x \ln^n x - n \int \ln^{n-1} x dx \\
 &= x \ln^n x - n x \ln^{n-1} x + n(n-1) \int \ln^{n-2} x dx = \dots
 \end{aligned}$$

$$= x[\ln^n x - n\ln^{n-1}x + n(n-1)\ln^{n-2}x - \dots \\ + (-1)^{n-1}n!\ln x + (-1)^nn!] + C.$$

2099. $\int x^3 \ln^3 x dx.$

解
$$\begin{aligned} \int x^3 \ln^3 x dx &= \frac{1}{4} \int \ln^3 x d(x^4) \\ &= \frac{1}{4} x^4 \ln^3 x - \frac{3}{4} \int x^3 \ln^2 x dx \\ &= \frac{1}{4} x^4 \ln^3 x - \frac{3}{16} \int \ln^2 x d(x^4) \\ &= \frac{1}{4} x^4 \ln^3 x - \frac{3}{16} x^4 \ln^2 x + \frac{3}{8} \int x^3 \ln x dx \\ &= \frac{1}{4} x^4 \ln^3 x - \frac{3}{16} x^4 \ln^2 x + \frac{3}{32} \int \ln x d(x^4) \\ &= \frac{1}{4} x^4 \ln^3 x - \frac{3}{16} x^4 \ln^2 x + \frac{3}{32} x^4 \ln x - \frac{3}{32} \int x^3 dx \\ &= \frac{1}{4} x^4 \left(\ln^3 x - \frac{3}{4} \ln^2 x + \frac{3}{8} \ln x - \frac{3}{32} \right) + C. \end{aligned}$$

2100. $\int \left(\frac{\ln x}{x} \right)^3 dx.$

解
$$\begin{aligned} \int \left(\frac{\ln x}{x} \right)^3 dx &= -\frac{1}{2} \int \ln^3 x d\left(\frac{1}{x^2} \right) \\ &= -\frac{1}{2x^2} \ln^3 x + \frac{3}{2} \int \frac{\ln^2 x}{x^3} dx \\ &= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4} \int \ln^2 x d\left(\frac{1}{x^2} \right) \\ &= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4x^2} \ln^2 x + \frac{3}{2} \int \frac{\ln x}{x^3} dx \\ &= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4x^2} \ln^2 x - \frac{3}{4} \int \ln x d\left(\frac{1}{x^2} \right) \\ &= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4x^2} \ln^2 x - \frac{3}{4x^2} \ln x + \frac{3}{4} \int \frac{dx}{x^3} \end{aligned}$$

$$= -\frac{1}{2x^2} \left(\ln^3 x + \frac{3}{2} \ln^2 x + \frac{3}{2} \ln x + \frac{3}{4} \right) + C.$$

$$2101. \int \ln[(x+a)^{x+a}(x+b)^{x+b}] \cdot \frac{dx}{(x+a)(x+b)}.$$

$$\begin{aligned} \text{解} \quad & \int \ln[(x+a)^{x+a}(x+b)^{x+b}] \cdot \frac{dx}{(x+a)(x+b)} \\ &= \int \frac{\ln(x+a)}{x+b} dx + \int \frac{\ln(x+b)}{x+a} dx \\ &= \int \ln(x+a) d[\ln(x+b)] \\ &\quad + \int \ln(x+b) d[\ln(x+a)] \\ &= \ln(x+a) \ln(x+b) - \int \ln(x+b) d[\ln(x+a)] \\ &\quad + \int \ln(x+b) d[\ln(x+a)] \\ &= \ln(x+a) \ln(x+b) + C. \end{aligned}$$

$$2102. \int \ln^2(x + \sqrt{1+x^2}) dx.$$

$$\begin{aligned} \text{解} \quad & \int \ln^2(x + \sqrt{1+x^2}) dx = x \ln^2(x + \sqrt{1+x^2}) \\ &\quad - 2 \int \frac{x}{\sqrt{1+x^2}} \ln(x + \sqrt{1+x^2}) dx \\ &= x \ln^2(x + \sqrt{1+x^2}) \\ &\quad - 2 \int \ln(x + \sqrt{1+x^2}) d(\sqrt{1+x^2}) \\ &= x \ln^2(x + \sqrt{1+x^2}) \\ &\quad - 2 \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2 \int dx \\ &= x \ln^2(x + \sqrt{1+x^2}) \\ &\quad - 2 \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2x + C. \end{aligned}$$

$$2103. \int \ln(\sqrt{1-x} + \sqrt{1+x}) dx.$$

$$\begin{aligned} \text{解} \quad & \int \ln(\sqrt{1-x} + \sqrt{1+x}) dx \\ &= x \ln(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2} \int \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} dx \\ &= x \ln(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2} \arcsin x - \frac{1}{2} x + C. \end{aligned}$$

$$2104. \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx = \int \ln x d\left(\frac{x}{\sqrt{1+x^2}}\right) \\ &= \frac{x \ln x}{\sqrt{1+x^2}} - \int \frac{dx}{\sqrt{1+x^2}} \\ &= \frac{x \ln x}{\sqrt{1+x^2}} - \ln(x + \sqrt{1+x^2}) + C. \end{aligned}$$

$$2105. \int x \operatorname{arctg}(x+1) dx.$$

$$\begin{aligned} \text{解} \quad & \int x \operatorname{arctg}(x+1) dx = \frac{1}{2} \int \operatorname{arctg}(x+1) d(x^2) \\ &= \frac{1}{2} x^2 \operatorname{arctg}(x+1) - \frac{1}{2} \int \frac{x^2}{x^2 + 2x + 2} dx \\ &= \frac{1}{2} x^2 \operatorname{arctg}(x+1) - \frac{1}{2} \int \left(1 - \frac{2x+2}{x^2 + 2x + 2}\right) dx \\ &= \frac{1}{2} x^2 \operatorname{arctg}(x+1) - \frac{x}{2} + \frac{1}{2} \ln(x^2 + 2x + 2) + C. \end{aligned}$$

$$2106. \int \sqrt{x} \operatorname{arctg} \sqrt{x} dx.$$

$$\begin{aligned} \text{解} \quad & \int \sqrt{x} \operatorname{arctg} \sqrt{x} dx = \frac{2}{3} \int \operatorname{arctg} \sqrt{x} d(x^{\frac{3}{2}}) \\ &= \frac{2}{3} x^{\frac{3}{2}} \operatorname{arctg} \sqrt{x} - \frac{1}{3} \int \frac{x}{1+x} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3} x^{\frac{3}{2}} \operatorname{arctg} \sqrt{x} - \frac{1}{3} \int \left(1 - \frac{1}{1+x} \right) dx \\
&= \frac{2}{3} x \sqrt{x} \operatorname{arctg} \sqrt{x} - \frac{x}{3} + \frac{1}{3} \ln(1+x) + C.
\end{aligned}$$

2107. $\int x \arcsin(1-x) dx$.

解 $\int x \arcsin(1-x) dx = \frac{1}{2} \int \arcsin(1-x) d(x^2)$
 $= \frac{1}{2} x^2 \arcsin(1-x) + \frac{1}{2} \int \frac{x^2}{\sqrt{1-(1-x)^2}} dx.$

对于积分 $\int \frac{x^2}{\sqrt{1-(1-x)^2}} dx$, 设 $1-x=t$, 则

$$\begin{aligned}
&\int \frac{x^2}{\sqrt{1-(1-x)^2}} dx = - \int \frac{1-2t+t^2}{\sqrt{1-t^2}} dt \\
&= \int \frac{-t^2+1}{\sqrt{1-t^2}} dt - 2 \int \frac{t}{\sqrt{1-t^2}} dt + 2 \int \frac{t dt}{\sqrt{1-t^2}} \\
&= \int \sqrt{1-t^2} dt - 2 \arcsin t - 2 \sqrt{1-t^2} \\
&= \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \arcsin t - 2 \arcsin t \\
&\quad - 2 \sqrt{1-t^2} + C_1 \\
&= \frac{-3-x}{2} \sqrt{2x-x^2} - \frac{3}{2} \arcsin(1-x) + C_1.
\end{aligned}$$

于是,

$$\begin{aligned}
&\int x \arcsin(1-x) dx \\
&= \frac{2x^2-3}{4} \arcsin(1-x) - \frac{3+x}{4} \sqrt{2x-x^2} + C.
\end{aligned}$$

2108. $\int \arcsin \sqrt{x} dx$.

解 $\int \arcsin \sqrt{x} dx = x \arcsin \sqrt{x}$

$$-\frac{1}{2} \int \frac{\sqrt{x}}{\sqrt{1-x}} dx.$$

对于积分 $\int \frac{\sqrt{x}}{\sqrt{1-x}} dx$, 设 $\sqrt{x} = t$, 则 $dx = 2t dt$.

于是,

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{1-x}} dx &= 2 \int \frac{t^2}{\sqrt{1-t^2}} dt \\ &= -2 \int \sqrt{1-t^2} dt + 2 \int \frac{dt}{\sqrt{1-t^2}} \\ &= -t \sqrt{1-t^2} - \arcsin t + 2 \arcsin t + C_1 \\ &= \arcsin \sqrt{x} - \sqrt{x-x^2} + C_1. \end{aligned}$$

因而,

$$\begin{aligned} \int \arcsin \sqrt{x} dx &= \left(x - \frac{1}{2} \right) \arcsin \sqrt{x} \\ &+ \frac{1}{2} \sqrt{x-x^2} + C. \end{aligned}$$

2109. $\int x \arccos \frac{1}{x} dx.$

$$\begin{aligned} \text{解} \quad \int x \arccos \frac{1}{x} dx &= \frac{1}{2} \int \arccos \frac{1}{x} d(x^2) \\ &= \frac{1}{2} x^2 \arccos \frac{1}{x} - \frac{1}{2} \int \frac{|x|}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2} x^2 \arccos \frac{1}{x} - \frac{1}{2} (\operatorname{sgn} x) \sqrt{x^2-1} + C. \end{aligned}$$

2110. $\int \arcsin \frac{2\sqrt{x}}{1+x} dx.$

$$\text{解} \quad \int \arcsin \frac{2\sqrt{x}}{1+x} dx = \int \arcsin \frac{2\sqrt{x}}{1+x} d(x+1)$$

$$2113. \int x \arctg x \ln(1+x^2) dx.$$

$$\begin{aligned}
 &\text{解} \quad \int x \arctg x \ln(1+x^2) dx \\
 &= \frac{1}{2} \int \arctg x \ln(1+x^2) d(x^2) \\
 &= \frac{1}{2} x^2 \arctg x \ln(1+x^2) \\
 &\quad - \frac{1}{2} \int x^2 \left[\frac{\ln(1+x^2)}{1+x^2} + \frac{2x \arctg x}{1+x^2} \right] dx \\
 &= \frac{1}{2} x^2 \arctg x \ln(1+x^2) - \frac{1}{2} \int \ln(1+x^2) dx \\
 &\quad + \frac{1}{2} \int \frac{\ln(1+x^2)}{1+x^2} dx + \int \frac{x \arctg x}{1+x^2} dx \\
 &\quad - \int x \arctg x dx \\
 &= \frac{1}{2} x^2 \arctg x \ln(1+x^2) \\
 &\quad - \frac{1}{2} x \ln(1+x^2) + \frac{1}{2} \int \frac{2x^2 dx}{1+x^2} \\
 &\quad + \frac{1}{2} \arctg x \ln(1+x^2) - \int \frac{x \arctg x}{1+x^2} dx \\
 &\quad + \int \frac{x \arctg x}{1+x^2} dx \\
 &\quad - \frac{1}{2} x^2 \arctg x + \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\
 &= \frac{1}{2} x^2 \arctg x \ln(1+x^2) - \frac{1}{2} x \ln(1+x^2) \\
 &\quad + x - \arctg x + \frac{1}{2} \arctg x \ln(1+x^2) \\
 &\quad - \frac{1}{2} x^2 \arctg x + \frac{1}{2} x - \frac{1}{2} \arctg x + C
 \end{aligned}$$

$$= x - \operatorname{arctg} x + \left(\frac{1+x^2}{2} \operatorname{arctg} x - \frac{x}{2} \right) \\ (\ln(1+x^2) - 1) + C.$$

2114. $\int x \ln \frac{1+x}{1-x} dx.$

解 $\int x \ln \frac{1+x}{1-x} dx = \frac{1}{2} \int \ln \frac{1+x}{1-x} d(x^2)$
 $= \frac{1}{2} x^2 \ln \frac{1+x}{1-x} - \int \frac{x^2}{1-x^2} dx$
 $= \frac{1}{2} x^2 \ln \frac{1+x}{1-x} + \int \left(1 - \frac{1}{1-x^2} \right) dx$
 $= \frac{x^2-1}{2} \ln \frac{1+x}{1-x} + x + C.$

2115. $\int \frac{\ln(x + \sqrt{1+x^2})}{(1+x^2)^{\frac{3}{2}}} dx.$

解 $\int \frac{\ln(x + \sqrt{1+x^2})}{(1+x^2)^{\frac{3}{2}}} dx$
 $= \int \ln(x + \sqrt{1+x^2}) d\left(\frac{x}{\sqrt{1+x^2}} \right)$
 $= \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} - \int \frac{x}{1+x^2} dx$
 $= \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} - \ln \sqrt{1+x^2} + C.$

求含有双曲线函数的积分:

2116. $\int \operatorname{sh}^2 x \operatorname{ch}^2 x dx.$

解 $\int \operatorname{sh}^2 x \operatorname{ch}^2 x dx = \frac{1}{4} \int \operatorname{sh}^2 2x dx$
 $= \frac{1}{8} \int \operatorname{sh}^2 2x d(2x)$

$$= -\frac{x}{8} + \frac{\text{sh}4x}{32} + C.$$

*) 利用 1761 题的结果.

2117. $\int \text{ch}^4 x dx$.

$$\begin{aligned} \text{解} \quad \int \text{ch}^4 x dx &= \int \left(\frac{1 + \text{ch}2x}{2} \right)^2 dx \\ &= \int \left(\frac{1}{4} + \frac{1}{2} \text{ch}2x + \frac{1}{4} \text{ch}^2 2x \right) dx \\ &= \frac{1}{4}x + \frac{1}{4} \text{sh}2x + \frac{1}{8} \left(x + \frac{1}{4} \text{sh}4x \right) + C \\ &= \frac{3}{8}x + \frac{1}{4} \text{sh}2x + \frac{1}{32} \text{sh}4x + C. \end{aligned}$$

*) 利用 1762 题的结果.

2118. $\int \text{sh}^3 x dx$.

$$\begin{aligned} \text{解} \quad \int \text{sh}^3 x dx &= \int \text{sh}^2 x \text{sh} x dx = \int (\text{ch}^2 x - 1) d(\text{ch} x) \\ &= \frac{1}{3} \text{ch}^3 x - \text{ch} x + C. \end{aligned}$$

2119. $\int \text{sh} x \text{sh} 2x \text{sh} 3x dx$.

$$\begin{aligned} \text{解} \quad \int \text{sh} x \text{sh} 2x \text{sh} 3x dx &= \int \frac{1}{2} (\text{ch} 4x - \text{ch} 2x) \text{sh} 2x dx \\ &= \frac{1}{2} \int \text{ch} 4x \text{sh} 2x dx - \frac{1}{2} \int \text{ch} 2x \text{sh} 2x dx \\ &= \frac{1}{4} \int (\text{sh} 6x - \text{sh} 2x) dx - \frac{1}{4} \int \text{sh} 4x dx \\ &= \frac{1}{24} \text{ch} 6x - \frac{1}{16} \text{ch} 4x - \frac{1}{8} \text{ch} 2x + C. \end{aligned}$$

$$2120. \int \operatorname{th} x dx.$$

$$\text{解} \quad \int \operatorname{th} x dx = \int \frac{\operatorname{sh} x}{\operatorname{ch} x} dx = \ln(\operatorname{ch} x) + C.$$

$$2121. \int \operatorname{cth}^2 x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{cth}^2 x dx &= \int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^2 x} dx = \int \frac{1 + \operatorname{sh}^2 x}{\operatorname{sh}^2 x} dx \\ &= x - \operatorname{cth} x + C. \end{aligned}$$

$$2122. \int \sqrt{\operatorname{th} x} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{\operatorname{th} x} dx &= \int \sqrt{\frac{e^x - e^{-x}}{e^x + e^{-x}}} dx \\ &= \int \frac{e^x - e^{-x}}{\sqrt{e^{2x} - e^{-2x}}} dx \\ &= \int \frac{e^{2x} dx}{\sqrt{e^{4x} - 1}} - \int \frac{e^{-2x} dx}{\sqrt{1 - e^{-4x}}} \\ &= \frac{1}{2} \int \frac{d(e^{2x})}{\sqrt{(e^{2x})^2 - 1}} + \frac{1}{2} \int \frac{d(e^{-2x})}{\sqrt{1 - (e^{-2x})^2}} \\ &= \frac{1}{2} \ln(e^{2x} + \sqrt{e^{4x} - 1}) + \frac{1}{2} \arcsin(e^{-2x}) + C. \end{aligned}$$

$$2123. \int \frac{dx}{\operatorname{sh} x + 2\operatorname{ch} x}.$$

解 设 $\operatorname{th} \frac{x}{2} = t$, 则

$$\operatorname{sh} x = \frac{2t}{1-t^2}, \quad \operatorname{ch} x = \frac{1+t^2}{1-t^2},$$

$$x = \ln \frac{1+t}{1-t}, \quad dx = \frac{2}{1-t^2} dt.$$

于是,

$$\begin{aligned}\int \frac{dx}{\operatorname{sh} x + 2\operatorname{ch} x} &= \int \frac{dt}{t^2 + t + 1} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t+1}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{1+2\operatorname{th} \frac{x}{2}}{\sqrt{3}} + C.\end{aligned}$$

2124. $\int \operatorname{sh} ax \sin bx dx$.

$$\begin{aligned}\text{解} \quad \int \operatorname{sh} ax \sin bx dx &= \frac{1}{2} \int e^{ax} \sin bx dx \\ &\quad - \frac{1}{2} \int e^{-ax} \sin bx dx \\ &= \frac{1}{2} e^{ax} \cdot \frac{a \sin bx - b \cos bx^{*)}}{a^2 + b^2} \\ &\quad + \frac{1}{2} e^{-ax} \cdot \frac{a \sin bx + b \cos bx^{*)}}{a^2 + b^2} + C \\ &= \frac{a \operatorname{ch} ax \cdot \sin bx - b \operatorname{sh} ax \cdot \cos bx}{a^2 + b^2} + C.\end{aligned}$$

*) 利用 1829 题的结果.

2125. $\int \operatorname{sh} ax \cos bx dx$.

$$\begin{aligned}\text{解} \quad \int \operatorname{sh} ax \cos bx dx &= \frac{1}{2} \int e^{ax} \cos bx dx \\ &\quad - \frac{1}{2} \int e^{-ax} \cos bx dx \\ &= \frac{1}{2} e^{ax} \cdot \frac{a \cos bx + b \sin bx^{*)}}{a^2 + b^2} \\ &\quad + \frac{1}{2} e^{-ax} \cdot \frac{a \cos bx - b \sin bx^{*)}}{a^2 + b^2} + C \\ &= \frac{a \operatorname{ch} ax \cdot \cos bx + b \operatorname{sh} ax \cdot \sin bx}{a^2 + b^2} + C.\end{aligned}$$

*) 利用 1828 题的结果.

§ 6. 函数的积分法的各种例子

求积分:

$$2126. \int \frac{dx}{x^6(1+x^2)}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x^6(1+x^2)} &= \int \frac{(x^2+1)-x^2}{x^6(1+x^2)} dx \\ &= \int \frac{dx}{x^6} - \int \frac{dx}{x^4(1+x^2)} \\ &= -\frac{1}{5x^5} - \int \frac{(x^2+1)-x^2}{x^4(1+x^2)} dx \\ &= -\frac{1}{5x^5} - \int \frac{dx}{x^4} - \int \frac{x^2}{x^4(1+x^2)} dx \\ &= -\frac{1}{5x^5} + \frac{1}{3x^3} + \int \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx \\ &= -\frac{1}{5x^5} + \frac{1}{3x^3} - \frac{1}{x} - \operatorname{arctg} x + C. \end{aligned}$$

$$2127. \int \frac{x^2 dx}{(1-x^2)^3}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2 dx}{(1-x^2)^3} &= \int \frac{(x^2-1)+1}{(1-x^2)^3} dx \\ &= -\int \frac{dx}{(x^2-1)^2} - \int \frac{dx}{(x^2-1)^3} \\ &= -\int \frac{dx}{(x^2-1)^2} - \left[\frac{2x}{2(-4)(x^2-1)^2} \right. \\ &\quad \left. - \frac{3}{4} \int \frac{dx}{(x^2-1)^2} \right]'' \\ &= -\frac{1}{4} \int \frac{dx}{(x^2-1)^2} + \frac{x}{4(1-x^2)^2} \end{aligned}$$

$$= -\frac{1}{4} \left\{ -\frac{x}{2(x^2-1)} - \frac{1}{2} \int \frac{dx}{x^2-1} \right\} \\ + \frac{x}{4(1-x^2)^2} = \frac{x+x^3}{8(1-x^2)^2} - \frac{1}{16} \ln \left| \frac{1+x}{1-x} \right| \\ + C.$$

*) 利用 1921 题的递推公式.

2128. $\int \frac{dx}{1+x^4+x^8}.$

解 因为

$$1+x^4+x^8=(x^4+1)^2-x^4=(x^4+x^2+1)(x^4-x^2+1), \\ x^4+x^2+1=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1), \\ x^4-x^2+1=(x^2+1)^2-3x^2=(x^2+x\sqrt{3}+1) \\ (x^2-x\sqrt{3}+1),$$

所以

$$\frac{1}{1+x^4+x^8} = \frac{1}{2} \left(\frac{x^2+1}{x^4+x^2+1} - \frac{x^2-1}{x^4-x^2+1} \right), \\ \frac{x^2+1}{x^4+x^2+1} = \frac{1}{2} \left(\frac{1}{x^2+x+1} + \frac{1}{x^2-x+1} \right), \\ \frac{x^2-1}{x^4-x^2+1} = \frac{-\frac{1}{\sqrt{3}}x - \frac{1}{2}}{x^2+x\sqrt{3}+1} + \frac{\frac{1}{\sqrt{3}}x - \frac{1}{2}}{x^2-x\sqrt{3}+1}.$$

于是,

$$\int \frac{dx}{1+x^4+x^8} = \frac{1}{4} \int \frac{dx}{x^2+x+1} + \frac{1}{4} \int \frac{dx}{x^2-x+1} \\ + \frac{1}{4\sqrt{3}} \int \frac{2x+\sqrt{3}}{x^2+x\sqrt{3}+1} dx \\ - \frac{1}{4\sqrt{3}} \int \frac{2x-\sqrt{3}}{x^2-x\sqrt{3}+1} dx$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{3}} \left(\operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + \operatorname{arctg} \left(\frac{2x-1}{\sqrt{3}} \right) \right) \\
&\quad + \frac{1}{4\sqrt{3}} \{ \ln(x^2 + x\sqrt{3} + 1) \\
&\quad - \ln(x^2 - x\sqrt{3} + 1) \} + C_1 \\
&= -\frac{1}{2\sqrt{3}} \operatorname{arctg} \left(\frac{1-x^2}{x\sqrt{3}} \right) \\
&\quad + \frac{1}{4\sqrt{3}} \ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + C.
\end{aligned}$$

2129. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$

解 设 $\sqrt[6]{x} = t$, 则

$$\sqrt{x} = t^3, \sqrt[3]{x} = t^2, dx = 6t^5 dt.$$

代入得

$$\begin{aligned}
\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= 6 \int \frac{t^3 dt}{t + 1} \\
&= 6 \int \left(t^2 - t + 1 - \frac{1}{t + 1} \right) dt \\
&= 2t^3 - 3t^2 + 6t - 6\ln(1 + t) + C \\
&= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1 + \sqrt[6]{x}) \\
&\quad + C (x > 0).
\end{aligned}$$

2130. $\int x^2 \sqrt{\frac{x}{1-x}} dx.$

解 设 $\sqrt{\frac{1-x}{x}} = t$, 则

$$x = \frac{1}{1+t^2} \quad dx = -\frac{2t}{(1+t^2)^2} dt.$$

代入得

$$\begin{aligned}
 \int x^2 \sqrt{\frac{x}{1-x}} dx &= -2 \int \frac{dt}{(t^2+1)^4} \\
 &= -2 \left[\frac{t}{6(t^2+1)^3} + \frac{5t}{24(t^2+1)^2} \right. \\
 &\quad \left. + \frac{5t}{16(t^2+1)} + \frac{5}{16} \operatorname{arctg} t \right]^{**} + C_1 \\
 &= -\frac{1}{24} (8x^2 + 10x + 15) \sqrt{x(1-x)} \\
 &\quad - \frac{5}{8} \operatorname{arctg} \sqrt{\frac{1-x}{x}} + C_1 \\
 &= -\frac{1}{24} (8x^2 + 10x + 15) \sqrt{x(1-x)} \\
 &\quad + \frac{5}{8} \arcsin \sqrt{x} + C (0 < x < 1).
 \end{aligned}$$

*) 利用 1921 题的递推公式.

2131. $\int \frac{x+2}{x^2 \sqrt{1-x^2}} dx.$

解 设 $x = \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则 $dx = \cos t dt$.

代入得

$$\begin{aligned}
 \int \frac{x+2}{x^2 \sqrt{1-x^2}} dx &= \int \frac{\sin t + 2}{\sin^2 t} dt \\
 &= \int \frac{dt}{\sin t} + 2 \int \frac{dt}{\sin^2 t} \\
 &= \ln |\csc t - \operatorname{ctg} t| - 2 \operatorname{ctg} t + C \\
 &= -\ln \frac{1 + \sqrt{1-x^2}}{|x|} - \frac{2\sqrt{1-x^2}}{x} + C (0 < |x|
 \end{aligned}$$

$$\begin{aligned}
& \int \frac{dx}{\sqrt[3]{x^2(1-x)}} = -3 \int \frac{t}{t^3+1} dt \\
&= \int \frac{dt}{t+1} - \int \frac{t+1}{t^2-t+1} dt \\
&= \ln|t+1| - \frac{1}{2} \int \frac{2t-1}{t^2-2t+1} dt - \frac{3}{2} \int \frac{dt}{t^2-t+1} \\
&= \frac{1}{2} \ln \frac{(t+1)^2}{t^2-t+1} - \sqrt{3} \operatorname{arctg} \left(\frac{2t-1}{\sqrt{3}} \right) + C,
\end{aligned}$$

其中 $t = \sqrt[3]{\frac{1-x}{x}}$.

2135. $\int \frac{dx}{x \sqrt{1+x^3+x^6}}.$

解
$$\begin{aligned}
\int \frac{dx}{x \sqrt{1+x^3+x^6}} &= \int \frac{dx}{x^4 \sqrt{x^{-6}+x^{-3}+1}} \\
&= -\frac{1}{3} \int \frac{d\left(x^{-3} + \frac{1}{2}\right)}{\sqrt{\left(x^{-3} + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\
&= -\frac{1}{3} \ln \left| x^{-3} + \frac{1}{2} + \sqrt{x^{-6}+x^{-3}+1} \right| + C_1 \\
&= -\frac{1}{3} \ln \left| \frac{2+x^3+2\sqrt{x^6+x^3+1}}{x^3} \right| + C.
\end{aligned}$$

注 以上实际已设 $x > 0$. 对于 $x < 0$, 利用 1856 题的方法可得同一结果.

2136. $\int \frac{dx}{x \sqrt{x^4-2x^2-1}}.$

解
$$\begin{aligned}
\int \frac{dx}{x \sqrt{x^4-2x^2-1}} &= \int \frac{dx}{x^3 \sqrt{1-2x^{-2}-x^{-4}}} \\
&= -\frac{1}{2} \int \frac{d(x^{-2}+1)}{\sqrt{2-(x^{-2}+1)^2}}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \arcsin\left(\frac{x^{-2}+1}{\sqrt{2}}\right) + C_1 \\
&= -\frac{1}{2} \arcsin\left(\frac{x^2+1}{x^2\sqrt{2}}\right) + C_1 \\
&= \frac{1}{2} \arccos\left(\frac{x^2+1}{x^2\sqrt{2}}\right) + C.
\end{aligned}$$

2137. $\int \frac{1 + \sqrt{1-x^2}}{1 - \sqrt{1-x^2}} dx.$

解
$$\begin{aligned}
&\int \frac{1 + \sqrt{1-x^2}}{1 - \sqrt{1-x^2}} dx \\
&= \int \frac{(1 + \sqrt{1-x^2})(1 + \sqrt{1-x^2})}{(1 - \sqrt{1-x^2})(1 + \sqrt{1-x^2})} dx \\
&= \int \frac{2 - x^2 + 2\sqrt{1-x^2}}{x^2} dx \\
&= -\frac{2}{x} - x - 2 \int \sqrt{1-x^2} d\left(\frac{1}{x}\right) \\
&= -\frac{2}{x} - x - \frac{2}{x} \sqrt{1-x^2} - 2 \int \frac{dx}{\sqrt{1-x^2}} \\
&= -\frac{2+x^2}{x} - \frac{2}{x} \sqrt{1-x^2} - 2 \arcsin x + C.
\end{aligned}$$

2138. $\int \frac{(1+x)dx}{x + \sqrt{x+x^2}}.$

解
$$\begin{aligned}
&\int \frac{(1+x)dx}{x + \sqrt{x+x^2}} \\
&= \int \frac{(1+x)(x - \sqrt{x+x^2})}{(x + \sqrt{x+x^2})(x - \sqrt{x+x^2})} dx \\
&= \int \frac{x + x^2 - \sqrt{x+x^2} - x\sqrt{x+x^2}}{-x} dx
\end{aligned}$$

$$\begin{aligned}
&= -x - \frac{1}{2}x^2 + \int \frac{\sqrt{1+x}}{\sqrt{x}} dx + \int \sqrt{x+x^2} dx \\
&= -x - \frac{1}{2}x^2 + 2 \int \sqrt{1+(\sqrt{x})^2} d(\sqrt{x}) \\
&\quad + \int \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} d\left(x + \frac{1}{2}\right) \\
&= -x - \frac{1}{2}x^2 + \sqrt{x} \cdot \sqrt{1+x} + \ln(\sqrt{x} \\
&\quad + \sqrt{1+x}) + \frac{2x+1}{4} \sqrt{x+x^2} \\
&\quad - \frac{1}{8} \ln\left(x + \frac{1}{2} + \sqrt{x+x^2}\right) + C_1 \\
&= -x - \frac{1}{2}x^2 + \frac{5+2x}{4} \sqrt{x+x^2} \\
&\quad + \frac{1}{2} \ln(2x+1+2\sqrt{x+x^2}) \\
&\quad - \frac{1}{8} \ln\left(x + \frac{1}{2} + \sqrt{x+x^2}\right) + C_1 \\
&= -\frac{1}{2}(x+1)^2 + \frac{5+2x}{4} \sqrt{x+x^2} \\
&\quad + \frac{3}{8} \ln\left(x + \frac{1}{2} + \sqrt{x+x^2}\right) + C,
\end{aligned}$$

其中设 $x > 0$, 对于 $x < -1$, 同样可获得上述结果, 但要注意加对数中的绝对值.

2139. $\int \frac{\ln(1+x+x^2)}{(1+x)^2} dx.$

解 $\int \frac{\ln(1+x+x^2)}{(1+x)^2} dx$

$$= - \int \ln(1+x+x^2) d\left(\frac{1}{1+x}\right)$$

$$\begin{aligned}
&= -\frac{\ln(1+x+x^2)}{1+x} + \int \frac{2x+1}{(x+1)(1+x+x^2)} dx \\
&= -\frac{\ln(1+x+x^2)}{1+x} + \int \left(\frac{x+2}{1+x+x^2} \right. \\
&\quad \left. - \frac{1}{1+x} \right) dx \\
&= -\frac{\ln(1+x+x^2)}{1+x} + \frac{1}{2} \int \left(\frac{2x+1}{1+x+x^2} \right. \\
&\quad \left. + \frac{3}{1+x+x^2} \right) dx - \ln|1+x| \\
&= -\frac{\ln(1+x+x^2)}{1+x} + \frac{1}{2} \ln(1+x+x^2) \\
&\quad + \sqrt{3} \operatorname{arc\,tg} \left(\frac{2x+1}{\sqrt{3}} \right) - \ln|1+x| + C \\
&= -\frac{\ln(1+x+x^2)}{1+x} - \frac{1}{2} \ln \frac{(1+x)^2}{1+x+x^2} \\
&\quad + \sqrt{3} \operatorname{arc\,tg} \left(\frac{2x+1}{\sqrt{3}} \right) + C.
\end{aligned}$$

2140. $\int (2x+3) \operatorname{arc\,cos}(2x-3) dx.$

$$\begin{aligned}
&\text{解} \quad \int (2x+3) \operatorname{arc\,cos}(2x-3) dx \\
&= \int \operatorname{arc\,cos}(2x-3) d(x^2+3x) \\
&= (x^2+3x) \operatorname{arccos}(2x-3) \\
&\quad + \int \frac{x^2+3x}{\sqrt{-x^2+3x-2}} dx \\
&= (x^2+3x) \operatorname{arccos}(2x-3) \\
&\quad - \int \sqrt{-x^2+3x-2} dx
\end{aligned}$$

$$\begin{aligned}
& - 3 \int \frac{-2x+3}{\sqrt{-x^2+3x-2}} dx \\
& + 7 \int \frac{dx}{\sqrt{-x^2+3x-2}} \\
& = (x^2+3x) \arccos(2x-3) \\
& - \int \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2} d\left(x - \frac{3}{2}\right) \\
& - 6 \sqrt{-x^2+3x-2} + 7 \int \frac{d\left(x - \frac{3}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} \\
& = (x^2+3x) \arccos(2x-3) \\
& - \frac{2x-3}{4} \sqrt{-x^2+3x-2} \\
& - \frac{1}{8} \arcsin(2x-3) - 6 \sqrt{-x^2+3x-2} \\
& - 7 \arccos(2x-3) + C_1 \\
& = \left(x^2+3x - \frac{55}{8}\right) \arccos(2x-3) \\
& - \frac{2x+21}{4} \sqrt{-x^2+3x-2} + C \quad (1 < x < 2).
\end{aligned}$$

2141. $\int x \ln(4+x^4) dx.$

$$\begin{aligned}
\text{解} \quad & \int x \ln(4+x^4) dx = \frac{1}{2} \int \ln(4+x^4) d(x^2) \\
& = \frac{1}{2} x^2 \ln(4+x^4) - 2 \int \frac{x^5}{4+x^4} dx \\
& = \frac{1}{2} x^2 \ln(4+x^4) - 2 \int \left(x - \frac{4x}{4+x^4}\right) dx \\
& = \frac{1}{2} x^2 \ln(4+x^4) - x^2 + 2 \operatorname{arctg}\left(\frac{x^2}{2}\right) + C.
\end{aligned}$$

$$2142. \int \frac{\arcsin x}{x^2} \cdot \frac{1+x^2}{\sqrt{1-x^2}} dx.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\arcsin x}{x^2} \cdot \frac{1+x^2}{\sqrt{1-x^2}} dx \\
 &= \int \frac{\arcsin x}{x^2 \sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx \\
 &= (\operatorname{sgn} x) \int \frac{\arcsin x dx}{x^3 \sqrt{x^{-2}-1}} + \int \arcsin x d(\arcsin x) \\
 &= -(\operatorname{sgn} x) \int \arcsin x d(\sqrt{x^{-2}-1}) \\
 &\quad + \frac{1}{2} (\arcsin x)^2 \\
 &= -(\operatorname{sgn} x) \cdot \left(\left(\frac{\sqrt{1-x^2}}{|x|} \arcsin x - \int \frac{dx}{|x|} \right) \right. \\
 &\quad \left. + \frac{1}{2} (\arcsin x)^2 \right) \\
 &= -\frac{\sqrt{1-x^2}}{x} \arcsin x + \int \frac{dx}{x} + \frac{1}{2} (\arcsin x)^2 \\
 &= -\frac{\sqrt{1-x^2}}{x} \arcsin x + \ln|x| \\
 &\quad + \frac{1}{2} (\arcsin x)^2 + C \quad (0 < |x| < 1).
 \end{aligned}$$

$$2143. \int \frac{x \ln(1 + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{x \ln(1 + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx \\
 &= \int \ln(1 + \sqrt{1+x^2}) d(1 + \sqrt{1+x^2}) \\
 &= (1 + \sqrt{1+x^2}) \ln(1 + \sqrt{1+x^2}) - \int \frac{x dx}{\sqrt{1+x^2}}
 \end{aligned}$$

$$= (1 + \sqrt{1+x^2}) \ln(1 + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

2144. $\int x \sqrt{x^2+1} \ln \sqrt{x^2-1} dx.$

解
$$\begin{aligned} & \int x \sqrt{x^2+1} \ln \sqrt{x^2-1} dx \\ &= \frac{1}{3} \int \ln \sqrt{x^2-1} d[(x^2+1)^{\frac{3}{2}}] \\ &= \frac{1}{3} (x^2+1)^{\frac{3}{2}} \ln \sqrt{x^2-1} \\ &\quad - \frac{1}{3} \int (x^2+1)^{\frac{3}{2}} \cdot \frac{x}{x^2-1} dx, \end{aligned}$$

对于右端的积分, 设 $\sqrt{x^2+1} = t$, 则 $x^2+1 = t^2$, $xdx = tdt$. 于是,

$$\begin{aligned} & -\frac{1}{3} \int (x^2+1)^{\frac{3}{2}} \frac{xdx}{x^2-1} = -\frac{1}{3} \int \frac{t^4 dt}{t^2-2} \\ &= -\frac{1}{3} \int \left(t^2 + 2 + \frac{4}{t^2-2} \right) dt \\ &= -\frac{1}{9} t^3 - \frac{2}{3} t - \frac{\sqrt{2}}{3} \ln \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + C \\ &= -\frac{x^2+7}{9} \sqrt{1+x^2} - \frac{\sqrt{2}}{3} \ln \frac{\sqrt{1+x^2} - \sqrt{2}}{\sqrt{1+x^2} + \sqrt{2}} + C. \end{aligned}$$

最后得到

$$\begin{aligned} & \int x \sqrt{x^2+1} \ln \sqrt{x^2-1} dx \\ &= \frac{1}{3} (x^2+1)^{\frac{3}{2}} \ln \sqrt{x^2-1} - \frac{x^2+7}{9} \sqrt{1+x^2} \\ &\quad - \frac{\sqrt{2}}{3} \ln \frac{\sqrt{1+x^2} - \sqrt{2}}{\sqrt{1+x^2} + \sqrt{2}} + C \quad (|x| > 1). \end{aligned}$$

$$2145^+. \int \frac{x}{\sqrt{1-x^2}} \ln \frac{x}{\sqrt{1-x}} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{x}{\sqrt{1-x^2}} \ln \frac{x}{\sqrt{1-x}} dx \\ &= - \int \ln \frac{x}{\sqrt{1-x}} d(\sqrt{1-x^2}) \\ &= - \sqrt{1-x^2} \ln \frac{x}{\sqrt{1-x}} \\ &\quad + \frac{1}{2} \int \frac{\sqrt{1-x^2}(2-x)}{x(1-x)} dx. \end{aligned}$$

右端的积分

$$\begin{aligned} & \int \frac{\sqrt{1-x^2}(2-x)}{x(1-x)} dx \\ &= \int \frac{(1-x^2)(2-x)}{x(1-x)\sqrt{1-x^2}} dx \\ &= \int \frac{2+x-x^2}{x\sqrt{1-x^2}} dx \\ &= 2 \int \frac{dx}{x\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x dx}{\sqrt{1-x^2}} \\ &= -2 \int \frac{d\left(\frac{1}{x}\right)}{\sqrt{\left(\frac{1}{x}\right)^2 - 1}} + \arcsin x + \sqrt{1-x^2} \\ &= -2 \ln \left| \frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right| + \arcsin x + \sqrt{1-x^2} \\ &\quad + C_1 \\ &= -2 \ln \frac{1 + \sqrt{1-x^2}}{x} + \arcsin x + \sqrt{1-x^2} + C_1. \end{aligned}$$

于是,

$$\begin{aligned} & \int \frac{x}{\sqrt{1-x^2}} \ln \frac{x}{\sqrt{1-x}} dx \\ &= \left(\frac{1}{2} - \ln \frac{x}{\sqrt{1-x}} \right) \sqrt{1-x^2} - \ln \frac{1+\sqrt{1-x^2}}{x} \\ &+ \frac{1}{2} \arcsin x + C \quad (0 < x < 1). \end{aligned}$$

2146. $\int \frac{dx}{(2+\sin x)^2}$

解 设 $\operatorname{tg} \frac{x}{2} = t$, 不妨限制 $-\pi < x < \pi$, 则

$$\sin x = \frac{2t}{1+t^2}, dx = \frac{2dt}{1+t^2}.$$

代入得

$$\begin{aligned} \int \frac{dx}{(2+\sin x)^2} &= \frac{1}{2} \int \frac{1+t^2}{(1+t+t^2)^2} dt \\ &= \frac{1}{2} \int \frac{(1+t+t^2) - \frac{1}{2}(2t+1) + \frac{1}{2}}{(1+t+t^2)^2} dt \\ &= \frac{1}{2} \int \frac{dt}{1+t+t^2} - \frac{1}{4} \int \frac{(2t+1)dt}{(1+t+t^2)^2} \\ &+ \frac{1}{4} \int \frac{dt}{(1+t+t^2)^2} \\ &= \frac{1}{\sqrt{3}} \operatorname{arctg} \left(\frac{2t+1}{\sqrt{3}} \right) + \frac{1}{4(1+t+t^2)} \\ &+ \frac{1}{4} \left(\frac{2t+1}{3(1+t+t^2)} + \frac{4}{3\sqrt{3}} \operatorname{arctg} \left(\frac{2t+1}{\sqrt{3}} \right) \right)^{**} \\ &+ C, \\ &= \frac{4}{3\sqrt{3}} \operatorname{arctg} \left(\frac{1+2\operatorname{tg} \frac{x}{2}}{\sqrt{3}} \right) + \frac{\cos x}{3(2+\sin x)}^{***} + C. \end{aligned}$$

*) 利用 1921 题的递推公式.

$$\begin{aligned}
&= 32 \cdot \frac{1}{8\sqrt{2}} \left(\int \frac{\sin 4x}{\cos 4x + 7 - 4\sqrt{2}} dx \right. \\
&\quad \left. - \int \frac{\sin 4x}{\cos 4x + 7 + 4\sqrt{2}} dx \right) \\
&= -\frac{1}{\sqrt{2}} \int \frac{d(\cos 4x + 7 - 4\sqrt{2})}{\cos 4x + 7 - 4\sqrt{2}} \\
&\quad + \frac{1}{\sqrt{2}} \int \frac{d(\cos 4x + 7 + 4\sqrt{2})}{\cos 4x + 7 + 4\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \ln \frac{\cos 4x + 7 + 4\sqrt{2}}{\cos 4x + 7 - 4\sqrt{2}} + C.
\end{aligned}$$

2148. $\int \frac{dx}{\sin x \sqrt{1 + \cos x}}.$

解 设 $1 + \cos x = t^2$, 并限制 $t > 0$, 则

$$\sin x = t \sqrt{2 - t^2}, dx = -\frac{2}{\sqrt{2 - t^2}} dt.$$

于是,

$$\begin{aligned}
\int \frac{dx}{\sin x \sqrt{1 + \cos x}} &= -\int \frac{2dt}{t^2(2 - t^2)} \\
&= -\int \left(\frac{1}{t^2} + \frac{1}{2 - t^2} \right) dt \\
&= \frac{1}{t} - \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2} + t}{\sqrt{2} - t} + C \\
&= \frac{1}{\sqrt{1 + \cos x}} - \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sqrt{2} - \sqrt{1 + \cos x}} + C.
\end{aligned}$$

2149. $\int \frac{ax^2 + b}{x^2 + 1} \operatorname{arctg} x dx.$

解 $\int \frac{ax^2 + b}{x^2 + 1} \operatorname{arctg} x dx = \int \left(a - \frac{a - b}{x^2 + 1} \right) \operatorname{arctg} x dx$

$$\begin{aligned}
&= ax \operatorname{arctg} x - a \int \frac{x dx}{1+x^2} - \frac{a-b}{2} (\operatorname{arctg} x)^2 \\
&= a \left(x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) \right) - \frac{a-b}{2} (\operatorname{arctg} x)^2 \\
&\quad + C.
\end{aligned}$$

2150. $\int \frac{ax^2+b}{x^2-1} \ln \left| \frac{x-1}{x+1} \right| dx.$

解
$$\begin{aligned}
&\int \frac{ax^2+b}{x^2-1} \ln \left| \frac{x-1}{x+1} \right| dx \\
&= \int \left(a + \frac{a+b}{x^2-1} \right) \ln \left| \frac{x-1}{x+1} \right| dx \\
&= ax \ln \left| \frac{x-1}{x+1} \right| - a \int \frac{2x dx}{x^2-1} \\
&\quad + \frac{a+b}{2} \int \ln \left| \frac{x-1}{x+1} \right| d \left(\ln \left| \frac{x-1}{x+1} \right| \right) \\
&= a \left(x \ln \left| \frac{x-1}{x+1} \right| - \ln |x^2-1| \right) \\
&\quad + \frac{a+b}{4} \ln^2 \left| \frac{x-1}{x+1} \right| + C.
\end{aligned}$$

2151. $\int \frac{x \ln x}{(1+x^2)^2} dx.$

解
$$\begin{aligned}
&\int \frac{x \ln x}{(1+x^2)^2} dx = -\frac{1}{2} \int \ln x d \left(\frac{1}{1+x^2} \right) \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{x(1+x^2)} \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \ln x - \frac{1}{4} \ln(1+x^2) + C \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{4} \ln \frac{x^2}{1+x^2} + C.
\end{aligned}$$

$$2152. \int \frac{x \operatorname{arctg} x}{\sqrt{1+x^2}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x \operatorname{arctg} x}{\sqrt{1+x^2}} dx &= \int \operatorname{arctg} x d(\sqrt{1+x^2}) \\ &= \sqrt{1+x^2} \operatorname{arctg} x - \int \frac{dx}{\sqrt{1+x^2}} \\ &= \sqrt{1+x^2} \operatorname{arctg} x - \ln(x + \sqrt{1+x^2}) + C. \end{aligned}$$

$$2153^+. \int \frac{\sin 2x dx}{\sqrt{1+\cos^4 x}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin 2x dx}{\sqrt{1+\cos^4 x}} &= - \int \frac{d(1+\cos 2x)}{\sqrt{(1+\cos 2x)^2+4}} \\ &= - \ln(1+\cos 2x + \sqrt{(1+\cos 2x)^2+4}) + C, \\ &= - \ln(\cos^2 x + \sqrt{1+\cos^4 x}) + C. \end{aligned}$$

$$2154. \int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx &= - \int x^2 \arccos x d(\sqrt{1-x^2}) \\ &= - x^2 \sqrt{1-x^2} \arccos x \\ &\quad + \int \sqrt{1-x^2} \left(2x \arccos x - \frac{x^2}{\sqrt{1-x^2}} \right) dx \\ &= - x^2 \sqrt{1-x^2} \arccos x \\ &\quad - \frac{2}{3} \int \arccos x d[(1-x^2)^{\frac{3}{2}}] - \int x^2 dx \\ &= - x^2 \sqrt{1-x^2} \arccos x - \frac{2}{3} (1-x^2)^{\frac{3}{2}} \arccos x \\ &\quad - \frac{2}{3} \int (1-x^2)^{\frac{3}{2}} \cdot \frac{dx}{\sqrt{1-x^2}} - \frac{1}{3} x^3 \\ &= - x^2 \sqrt{1-x^2} \arccos x - \frac{2}{3} (1-x^2)^{\frac{3}{2}} \arccos x \end{aligned}$$

$$\begin{aligned}
& -\frac{2}{3}x + \frac{2}{9}x^3 - \frac{1}{3}x^3 + C \\
& = -\frac{6x + x^3}{9} - \frac{2 + x^2}{3} \sqrt{1 - x^2} \arccos x + C.
\end{aligned}$$

2155. $\int \frac{x^4 \operatorname{arctg} x}{1 + x^2} dx.$

解
$$\begin{aligned}
\int \frac{x^4 \operatorname{arctg} x}{1 + x^2} dx &= \int \left(x^2 - 1 + \frac{1}{x^2 + 1} \right) \operatorname{arctg} x dx \\
&= \frac{1}{3} \int \operatorname{arctg} x d(x^3) - \int \operatorname{arctg} x dx \\
&\quad + \int \operatorname{arctg} x d(\operatorname{arctg} x) \\
&= \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{3} \int \frac{x^3 dx}{1 + x^2} - x \operatorname{arctg} x \\
&\quad + \int \frac{x dx}{1 + x^2} + \frac{1}{2} (\operatorname{arctg} x)^2 \\
&= \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{3} \int \left(x - \frac{x}{1 + x^2} \right) dx - x \operatorname{arctg} x \\
&\quad + \frac{1}{2} \ln(1 + x^2) + \frac{1}{2} (\operatorname{arctg} x)^2 \\
&= \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1 + x^2) - x \operatorname{arctg} x \\
&\quad + \frac{1}{2} \ln(1 + x^2) + \frac{1}{2} (\operatorname{arctg} x)^2 + C \\
&= -\frac{1}{6} x^2 - \left(x - \frac{x^3}{3} \right) \operatorname{arctg} x \\
&\quad + \frac{1}{2} (\operatorname{arctg} x)^2 + \frac{2}{3} \ln(1 + x^2) + C.
\end{aligned}$$

2156. $\int \frac{x \operatorname{arctg} x}{(1 + x^2)^2} dx.$

解
$$\int \frac{x \operatorname{arctg} x}{(1 + x^2)^2} dx = -\frac{1}{2} \int \operatorname{arctg} x d\left(\frac{1}{1 + x^2} \right)$$

$$\begin{aligned}
&= -\frac{\operatorname{arccotg} x}{2(1+x^2)} - \frac{1}{2} \int \frac{dx}{(1+x^2)^2} \\
&= -\frac{\operatorname{arccotg} x}{2(1+x^2)} - \frac{1}{2} \left(\frac{x}{2(x^2+1)} - \frac{1}{2} \operatorname{arccotg} x \right) + C \\
&= -\frac{1-x^2}{4(1+x^2)} \operatorname{arccotg} x - \frac{x}{4(1+x^2)} + C.
\end{aligned}$$

*) 利用 1921 题的递推公式.

$$2157^+. \int \frac{x \ln(x + \sqrt{1+x^2})}{(1-x^2)^2} dx.$$

$$\begin{aligned}
\text{解} \quad & \int \frac{x \ln(x + \sqrt{1+x^2})}{(1-x^2)^2} dx \\
&= \frac{1}{2} \int \ln(x + \sqrt{1+x^2}) d\left(\frac{1}{1-x^2}\right) \\
&= \frac{1}{2(1-x^2)} \ln(x + \sqrt{1+x^2}) \\
&\quad - \frac{1}{2} \int \frac{dx}{(1-x^2)\sqrt{x^2+1}}.
\end{aligned}$$

对于右端积分设 $x = \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$\sqrt{1+x^2} = \sec t, dx = \sec^2 t dt.$$

于是,

$$\begin{aligned}
& \int \frac{dx}{(1-x^2)\sqrt{1+x^2}} = \int \frac{\sec t dt}{1-\operatorname{tg}^2 t} \\
&= \int \frac{\cos t dt}{\cos^2 t - \sin^2 t} = \int \frac{d(\sin t)}{1-2\sin^2 t} \\
&= \frac{1}{2\sqrt{2}} \ln \left| \frac{1 + \sqrt{2} \sin t}{1 - \sqrt{2} \sin t} \right| + C \\
&= \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{1+x^2} + x\sqrt{2}}{\sqrt{1+x^2} - x\sqrt{2}} \right| + C,
\end{aligned}$$

因而,

$$\int \frac{x \ln(x + \sqrt{1+x^2})}{(1+x^2)^2} dx = \frac{\ln(x + \sqrt{1+x^2})}{2(1-x^2)} + \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{1+x^2} - x\sqrt{2}}{\sqrt{1+x^2} + x\sqrt{2}} \right| + C.$$

2158. $\int \sqrt{1-x^2} \arcsin x dx.$

解 $\int \sqrt{1-x^2} \arcsin x dx = x \sqrt{1-x^2} \arcsin x - \int x \left(1 - \frac{x}{\sqrt{1-x^2}} \arcsin x \right) dx$
 $= x \sqrt{1-x^2} \arcsin x - \frac{x^2}{2} - \int \sqrt{1-x^2} \arcsin x dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx$
 $= x \sqrt{1-x^2} \arcsin x - \frac{x^2}{2} + \frac{1}{2} (\arcsin x)^2 - \int \sqrt{1-x^2} \arcsin x dx,$

于是,

$$\int \sqrt{1-x^2} \arcsin x dx = \frac{x}{2} \sqrt{1-x^2} \arcsin x - \frac{x^2}{4} + \frac{1}{4} (\arcsin x)^2 + C \quad (|x| < 1).$$

2159. $\int x(1+x^2) \operatorname{arccotg} x dx.$

解 $\int x(1+x^2) \operatorname{arccotg} x dx = \frac{1}{4} \int \operatorname{arccotg} x d[(1+x^2)^2]$

$$\begin{aligned}
&= \frac{1}{4}(1+x^2)^2 \operatorname{arccotg} x + \frac{1}{4} \int (1+x^2) dx \\
&= \frac{1}{4}(1+x^2)^2 \operatorname{arccotg} x + \frac{x}{4} + \frac{x^3}{12} + C.
\end{aligned}$$

2160. $\int x^x (1 + \ln x) dx.$

解 $\int x^x (1 + \ln x) dx = \int e^{x \ln x} (1 + \ln x) dx$
 $= \int e^{x \ln x} d(x \ln x)$
 $= e^{x \ln x} + C = x^x + C (x > 0).$

2161. $\int \frac{\arcsin e^x}{e^x} dx.$

解 $\int \frac{\arcsin e^x}{e^x} dx = - \int \arcsin e^x d(e^{-x})$
 $= - e^{-x} \arcsin e^x + \int \frac{dx}{\sqrt{1-e^{2x}}}$
 $= - e^{-x} \arcsin e^x - \int \frac{d(e^{-x})}{\sqrt{(e^{-x})^2 - 1}}$
 $= - e^{-x} \arcsin e^x - \ln(e^{-x} + \sqrt{(e^{-x})^2 - 1}) + C$
 $= x - e^{-x} \arcsin e^x - \ln(1 + \sqrt{1-e^{2x}}) + C (x < 0).$

2162. $\int \frac{\operatorname{arctg} e^{\frac{x}{2}}}{e^{\frac{x}{2}}(1+e^x)} dx.$

解 $\int \frac{\operatorname{arctg} e^{\frac{x}{2}}}{e^{\frac{x}{2}}(1+e^x)} dx = \int \left(e^{-\frac{x}{2}} - \frac{e^{\frac{x}{2}}}{1+e^x} \right) \operatorname{arctg} e^{\frac{x}{2}} dx$
 $= - 2 \int \operatorname{arctg} e^{\frac{x}{2}} d(e^{-\frac{x}{2}})$
 $= - 2 \int \operatorname{arctg} e^{\frac{x}{2}} d(\operatorname{arctg} e^{\frac{x}{2}})$
 $= - 2 e^{-\frac{x}{2}} \operatorname{arctg} e^{\frac{x}{2}} + \int \frac{dx}{1+e^x} - (\operatorname{arctg} e^{\frac{x}{2}})^2$

$$\begin{aligned}
&= -2e^{-\frac{x}{2}} \operatorname{arctg} e^{\frac{x}{2}} + \int \left(1 - \frac{e^x}{1+e^x} \right) dx \\
&\quad - (\operatorname{arctg} e^{\frac{x}{2}})^2 \\
&= -2e^{-\frac{x}{2}} \operatorname{arctg} e^{\frac{x}{2}} + x - \ln(1+e^x) - (\operatorname{arctg} e^{\frac{x}{2}})^2 \\
&\quad + C.
\end{aligned}$$

2163. $\int \frac{dx}{(e^{x+1} + 1)^2 - (e^{x-1} + 1)^2}.$

解
$$\begin{aligned}
&\int \frac{dx}{(e^{x+1} + 1)^2 - (e^{x-1} + 1)^2} \\
&= \int \frac{dx}{(e^{x+1} - e^{x-1})(e^{x+1} + e^{x-1} + 2)} \\
&= \int \frac{dx}{e^{2x}(e - e^{-1})(e + e^{-1} + 2e^{-x})} \\
&= \int \frac{dx}{e^{2x} \cdot 2\operatorname{sh}1 \cdot (2\operatorname{ch}1 + 2e^{-x})} \\
&= \int \frac{dx}{4e^x \operatorname{sh}1 \cdot (1 + e^x \operatorname{ch}1)} \\
&= \frac{1}{4\operatorname{sh}1} \int \left(\frac{1}{e^x} - \frac{\operatorname{ch}1}{1 + e^x \operatorname{ch}1} \right) dx \\
&= -\frac{e^{-x}}{4\operatorname{sh}1} - \frac{\operatorname{ch}1}{4\operatorname{sh}1} \int \left(1 - \frac{e^x \operatorname{ch}1}{1 + e^x \operatorname{ch}1} \right) dx \\
&= -\frac{e^{-x}}{4\operatorname{sh}1} - \frac{\operatorname{cth}1}{4} [x - \ln(1 + e^x \operatorname{ch}1)] + C.
\end{aligned}$$

2164. $\int \sqrt{\operatorname{th}^2 x + 1} dx.$

解
$$\begin{aligned}
&\int \sqrt{\operatorname{th}^2 x + 1} dx = \int \frac{\operatorname{th}^2 x + 1}{\sqrt{\operatorname{th}^2 x + 1}} dx \\
&= \int \frac{\operatorname{sh}^2 x + \operatorname{ch}^2 x}{\operatorname{ch}^2 x \sqrt{\operatorname{th}^2 x + 1}} dx
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{2\operatorname{ch}^2 x - 1}{\sqrt{1 + \operatorname{th}^2 x}} d(\operatorname{th} x) \\
&= 2 \int \frac{\operatorname{ch}^2 x d(\operatorname{th} x)}{\sqrt{1 + \operatorname{th}^2 x}} - \int \frac{d(\operatorname{th} x)}{\sqrt{1 + \operatorname{th}^2 x}} \\
&= 2 \int \frac{dx}{\sqrt{\operatorname{th}^2 x + 1}} - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) \\
&= 2 \int \frac{\operatorname{ch} x dx}{\sqrt{\operatorname{sh}^2 x + \operatorname{ch}^2 x}} - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) \\
&= \sqrt{2} \int \frac{d(\sqrt{2} \operatorname{sh} x)}{\sqrt{1 + 2\operatorname{sh}^2 x}} - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) \\
&= \sqrt{2} \ln(\sqrt{2} \operatorname{sh} x + \sqrt{1 + 2\operatorname{sh}^2 x}) \\
&\quad - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) + C \\
&= \frac{1}{\sqrt{2}} \ln \frac{\sqrt{1 + \operatorname{th}^2 x} + \sqrt{2} \operatorname{th} x}{\sqrt{1 + \operatorname{th}^2 x} - \sqrt{2} \operatorname{th} x} \\
&\quad - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) + C.
\end{aligned}$$

2165. $\int \frac{1 + \sin x}{1 + \cos x} \cdot e^x dx.$

解 $\int \frac{1 + \sin x}{1 + \cos x} \cdot e^x dx$

$$\begin{aligned}
&= \int \left[\frac{1 + 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} \right] e^x dx \\
&= \int \frac{e^x}{2\cos^2 \frac{x}{2}} dx + \int e^x \operatorname{tg} \frac{x}{2} dx \\
&= \int e^x d\left(\operatorname{tg} \frac{x}{2}\right) + \int \operatorname{tg} \frac{x}{2} d(e^x) \\
&= e^x \operatorname{tg} \frac{x}{2} - \int \operatorname{tg} \frac{x}{2} de^x + \int \operatorname{tg} \frac{x}{2} d(e^x)
\end{aligned}$$

2170. $\int e^{-|x|} dx.$

解 当 $x \geq 0$ 时, $\int e^{-|x|} dx = \int e^{-x} dx = -e^{-x} + C_1,$

当 $x < 0$ 时, $\int e^{-|x|} dx = \int e^x dx = e^x + C_2.$

由于 $e^{-|x|}$ 在 $(-\infty, +\infty)$ 上连续, 故其原函数必在 $(-\infty, +\infty)$ 上连续可微, 而且任意两个原函数之间差一常数. 今求满足 $F(0) = 0$ 的原函数 $F(x)$. 由上述知, 必有

$$F(x) = \begin{cases} -e^{-x} + C_1, & x \geq 0, \\ e^x + C_2, & x < 0. \end{cases}$$

其中 C_1, C_2 是两个常数. 由于 $0 = F(0) = \lim_{x \rightarrow 0-} F(x)$, 即 $0 = -1 + C_1 = 1 + C_2$, 因此 $C_1 = 1, C_2 = -1$, 从而

$$F(x) = \begin{cases} 1 - e^{-x}, & x \geq 0; \\ e^x - 1, & x < 0. \end{cases}$$

所以,

$$\int e^{-|x|} dx = \begin{cases} 1 - e^{-x} + C, & x \geq 0; \\ e^x - 1 + C, & x < 0. \end{cases}$$

2171. $\int \max(1, x^2) dx.$

解 仿上题, 当 $|x| \leq 1$ 时,

$$\int \max(1, x^2) dx = \int dx = x + C_1;$$

当 $x > 1$ 时,

$$\int \max(1, x^2) dx = \int x^2 dx = \frac{1}{3}x^3 + C_2,$$

当 $x < -1$ 时,

$$\int \max(1, x^2) dx = \int x^2 dx = \frac{1}{3}x^3 + C_3.$$

今求满足 $F(1) = 1$ 的原函数 $F(x)$. 由上述知

$$F(x) = \begin{cases} x + C_1, & -1 \leq x \leq 1; \\ \frac{1}{3}x^3 + C_2, & x > 1; \\ \frac{1}{3}x^3 + C_3, & x < -1. \end{cases}$$

其中 C_1, C_2, C_3 是三个常数. 由于 $1 = F(1) = \lim_{x \rightarrow 1+0} F(x)$, 即 $1 = 1 + C_1 = \frac{1}{3} + C_2$, 故 $C_1 = 0, C_2 = \frac{2}{3}$.

再由 $F(-1) = \lim_{x \rightarrow -1-0} F(x)$, 得 $-1 = -\frac{1}{3} + C_3$, 故 $C_3 = -\frac{2}{3}$.

由此可知

$$F(x) = \begin{cases} x, & -1 \leq x \leq 1; \\ \frac{1}{3}x^3 + \frac{2}{3}, & x > 1; \\ \frac{1}{3}x^3 - \frac{2}{3}, & x < -1. \end{cases}$$

最后得

$$\begin{aligned} & \int \max(1, x^2) dx \\ &= \begin{cases} x + C, & |x| \leq 1; \\ \frac{x^3}{3} + \frac{2}{3} \operatorname{sgn} x + C, & |x| > 1. \end{cases} \end{aligned}$$

2172. $\int \varphi(x) dx$, 其中 $\varphi(x)$ 为数 x 至其最接近的整数之距离.

解 显然 $\varphi(x)$ 在 $(-\infty, +\infty)$ 上连续, 故其原函数

在 $(-\infty, +\infty)$ 上连续可微. 今求满足 $F(0) = 0$ 的原函数. 由于

$$\varphi(x) = \begin{cases} x - n, & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -x + n + 1, & \text{当 } n + \frac{1}{2} \leq x < n + 1 \text{ 时} \end{cases}$$

故

$$F(x) = \begin{cases} \frac{x^2}{2} - nx + C_n, & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -\frac{x^2}{2} + (n+1)x + C_n', & \text{当 } n + \frac{1}{2} \leq x < n + 1 \text{ 时.} \end{cases}$$

其中 C_n, C_n' 是两个常数. 由 $\lim_{x \rightarrow (n+\frac{1}{2})-0} F(x) = F(n + \frac{1}{2})$,

$$\text{得 } C_n' = C_n - \left(n + \frac{1}{2}\right)^2.$$

故

$$F(x) = \begin{cases} \frac{x^2}{2} - nx + C_n, & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -\frac{x^2}{2} + (n+1)x - \left(n + \frac{1}{2}\right)^2 + C_n, & \text{当 } n + \frac{1}{2} \leq x < n + 1 \text{ 时.} \end{cases}$$

$$\text{由 } \lim_{x \rightarrow (n+1)-0} F(x) = F(n+1)$$

$$\text{得递推公式 } C_{n+1} = C_n + n + \frac{3}{4}.$$

$$\text{显然 } 0 = F(0) = C_0. \text{ 由此得 } C_n = \frac{1}{4}n(2n+1).$$

于是

$$F(x) = \begin{cases} \frac{x^2}{2} - nx + \frac{1}{4}n(2n+1) = \frac{x}{4} \\ + \frac{1}{4} \left(x - n - \frac{1}{2} \right) \cdot \left[1 - 2 \left(\frac{1}{2} - x + n \right) \right], \\ \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -\frac{x^2}{2} + (n+1)x - \frac{1}{4}(2n+1)(n+1) = \frac{x}{4} \\ + \frac{1}{4} \cdot \left(x - n - \frac{1}{2} \right) \left[1 - 2 \left(x - n - \frac{1}{2} \right) \right], \\ \text{当 } n + \frac{1}{2} \leq x < n + 1 \text{ 时} \end{cases}$$

记 $\{x\} = x - [x]$ 表数 x 去掉其整数部分 $[x]$ 后所剩下的零头部分, 那么最后得 $F(x) = \frac{x}{4} + \frac{1}{4} \left(\{x\} - \frac{1}{2} \right) \cdot \left\{ 1 - 2 \left| \{x\} - \frac{1}{2} \right| \right\} (-\infty < x < +\infty)$.

故

$$\int \varphi(x) dx = \frac{x}{4} + \frac{1}{4} \left(\{x\} - \frac{1}{2} \right) \cdot \left\{ 1 - 2 \left| \{x\} - \frac{1}{2} \right| \right\} + C (-\infty < x < +\infty).$$

2173. $\int [x] |\sin \pi x| dx \quad (x \geq 0).$

解 分别求出在区间 $[0, 1)$, $[1, 2)$, $[2, 3)$, \dots , $[n, n+1)$ 上满足 $F(0) = 0$ 的原函数 $F(x)$ 的增量如下:

在 $[0, 1)$ 上, $\int 0 \cdot \sin \pi x dx = C_1, F(1) - F(0) = 0;$

在 $[1, 2)$ 上, $-\int \sin \pi x dx = \frac{1}{\pi} \cos \pi x + C_2, F(2)$

$$-F(1) = \frac{2}{\pi};$$

$$\text{在 } [2, 3) \text{ 上, } 2 \int \sin \pi x dx = -\frac{2}{\pi} \cos \pi x + C_3, F(3)$$

$$-F(2) = \frac{2 \cdot 2}{\pi}; \dots\dots$$

$$\text{在 } [x], x] \text{ 上, } (-1)^{[x]} [x] \int \sin \pi x dx = (-1)^{[x]} \\ \cdot [x] \left(-\frac{1}{\pi} \right) \cos \pi x + C_{[x]+1},$$

$$F(x) - F([x]) = \frac{(-1)^{[x]} [x]}{\pi} (\cos \pi [x] - \cos \pi x).$$

从而, 对于 $x \geq 0$, 得到

$$\begin{aligned} \int [x] |\sin \pi x| dx &= F(x) + C = (F(1) - F(0)) \\ &+ (F(2) - F(1)) + (F(3) - F(2)) + \dots \\ &+ \frac{(-1)^{[x]} [x]}{\pi} (\cos \pi [x] - \cos \pi x) + C \\ &= \frac{2}{\pi} + \frac{2 \cdot 2}{\pi} + \dots + \frac{2([x] - 1)}{\pi} \\ &+ \frac{(-1)^{[x]} [x]}{\pi} (\cos \pi [x] - \cos \pi x) + C \\ &= \frac{[x] \cdot ([x] - 1)}{\pi} + \frac{(-1)^{[x]} \cdot [x] \cdot (-1)^{[x]}}{\pi} \\ &- \frac{(-1)^{[x]} \cdot [x] \cdot \cos \pi x}{\pi} + C \\ &= \frac{[x]}{\pi} ([x] - (-1)^{[x]} \cos \pi x) + C. \end{aligned}$$

$$2174. \int f(x) dx, \text{ 其中 } f(x) = \begin{cases} 1 - x^2, & \text{当 } |x| \leq 1; \\ 1 - |x|, & \text{当 } |x| > 1. \end{cases}$$

解 当 $|x| \leq 1$ 时,

$$\int f(x)dx = \int (1 - x^2)dx = x - \frac{x^3}{3} + C_1;$$

$$\begin{aligned} \text{当 } x > 1 \text{ 时, } \int f(x)dx &= \int (1 - |x|)dx \\ &= x - \frac{x|x|}{2} + C_2; \end{aligned}$$

$$\begin{aligned} \text{当 } x < -1 \text{ 时, } \int f(x)dx &= \int (1 - |x|)dx \\ &= x - \frac{x|x|}{2} + C_3. \end{aligned}$$

今求满足 $F(0) = 0$ 的原函数 $F(x)$. 利用 $F(0) = 0$, $\lim_{x \rightarrow 1+0} F(x) = F(1)$, $\lim_{x \rightarrow -1-0} F(x) = F(-1)$, 仿 2171 题, 可得

$$F(x) = \begin{cases} x - \frac{x^3}{3}, & |x| \leq 1; \\ x - \frac{x|x|}{2} + \frac{1}{6}, & x > 1; \\ x - \frac{x|x|}{2} - \frac{1}{6}, & x < -1. \end{cases}$$

于是

$$\int f(x)dx = \begin{cases} x - \frac{x^3}{3} + C, & |x| \leq 1; \\ x - \frac{x|x|}{2} + \frac{1}{6}\operatorname{sgn}x + C, & |x| > 1. \end{cases}$$

2175. $\int f(x)dx$, 式中

$$f(x) = \begin{cases} 1, & \text{若 } -\infty < x < 0; \\ x+1, & \text{若 } 0 \leq x \leq 1; \\ 2x, & \text{若 } 1 < x < +\infty. \end{cases}$$

解 当 $-\infty < x < 0$ 时,

$$\int f(x)dx = \int dx = x + C_1;$$

当 $0 \leq x \leq 1$ 时,

$$\int f(x)dx = \int (x+1)dx = \frac{x^2}{2} + x + C_2;$$

当 $1 < x < +\infty$ 时,

$$\int f(x)dx = \int 2xdx = x^2 + C_3.$$

今求满足 $F(0) = 0$ 的原函数 $F(x)$. 利用 $F(0) = 0$, $\lim_{x \rightarrow 0-0} F(x) = F(0)$, $\lim_{x \rightarrow 1+0} F(x) = F(1)$, 仿 2171 题, 可得

$$F(x) = \begin{cases} x, & \text{当 } -\infty < x < 0 \text{ 时;} \\ \frac{x^2}{2} + x, & \text{当 } 0 \leq x \leq 1 \text{ 时;} \\ x^2 + \frac{1}{2}, & \text{当 } 1 < x < +\infty \text{ 时.} \end{cases}$$

故

$$\int f(x)dx = \begin{cases} x + C, & \text{当 } -\infty < x < 0 \text{ 时;} \\ \frac{x^2}{2} + x + C, & \text{当 } 0 \leq x \leq 1 \text{ 时;} \\ x^2 + \frac{1}{2} + C, & \text{当 } 1 < x < +\infty \text{ 时.} \end{cases}$$

2176. 求 $\int x f''(x) dx$.

$$\begin{aligned} \text{解} \quad \int x f''(x) dx &= \int x d[f'(x)] = x f'(x) \\ &\quad - \int f'(x) dx = x f'(x) - f(x) + C. \end{aligned}$$

2177. 求 $\int f'(2x) dx$.

解 $\int f'(2x)dx = \frac{1}{2} \int f'(2x)d(2x) = \frac{1}{2} f(2x) + C.$

* 这里暗中分别假定了被积函数 f, f' 是连续的.

2178. 设 $f'(x^2) = \frac{1}{x} (x > 0)$, 求 $f(x)$.

解 由 $f'(x^2) = \frac{1}{x}$, 得 $f'(x) = \frac{1}{\sqrt{x}} (x > 0)$.

于是,

$$f(x) = \int f'(x)dx = \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C.$$

2179. 设 $f'(\sin^2 x) = \cos^2 x$, 求 $f(x)$.

解 由 $f'(\sin^2 x) = \cos^2 x = 1 - \sin^2 x$ 得 $f'(x) = 1 - x$.

于是,

$$\begin{aligned} f(x) &= \int f'(x)dx = \int (1-x)dx \\ &= x - \frac{1}{2}x^2 + C (|x| \leq 1). \end{aligned}$$

2180. 设 $f'(\ln x) = \begin{cases} 1, & \text{当 } 0 < x \leq 1; \\ x, & \text{当 } 1 < x < +\infty \end{cases}$
及 $f(0) = 0$, 求 $f(x)$.

解 设 $t = \ln x$, 则

$$f'(t) = \begin{cases} 1, & -\infty < t \leq 0, \\ e^t, & 0 < t < +\infty. \end{cases}$$

于是,

$$f(x) = \int f'(x)dx = \begin{cases} x + C_1, & -\infty < x \leq 0; \\ e^x + C_2, & 0 < x < +\infty, \end{cases}$$

其中 C_1, C_2 是两个常数. 由假定 $f(0) = 0$, 得 $C_1 = 0$.

再由 $f(x)$ 在 $x = 0$ 的连续性, 知 $f(0) = \lim_{x \rightarrow 0+} f(x)$, 由此得 $C_2 = -1$.

于是

$$f(x) = \begin{cases} x, & \text{当 } -\infty < x \leq 0 \text{ 时;} \\ e^x - 1, & \text{当 } 0 < x < +\infty \text{ 时.} \end{cases}$$

$f(x)$ 于已知闭区间 $[a, b]$ 上可积分的充要条件.

2181. 把区间 $[-1, 4]$ 分为 n 个相等的子区间, 并取这些子区间中点的坐标作自变量 ξ 的值 ($i = 0, 1, \dots, n-1$). 求函数 $f(x) = 1 + x$ 在此区间上的积分和 S_n .

解 每一个子区间的长为 $\frac{5}{n}$, 第 i 个子区间为 $(-1 + \frac{5i}{n}, -1 + \frac{5i}{n} + \frac{5}{n})$, 其中点 $\xi_i = -1 + (i + \frac{1}{2}) \cdot \frac{5}{n}$. 于是, 所求的积分和为

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} \left\{ 1 + \left[-1 + \left(i + \frac{1}{2} \right) \frac{5}{n} \right] \right\} \cdot \frac{5}{n} \\ &= \frac{25}{n^2} \sum_{i=0}^{n-1} \left(i + \frac{1}{2} \right) = 12 \frac{1}{2}. \end{aligned}$$

2182. 设

$$(a) \quad f(x) = x^3 \quad [-2 \leq x \leq 3];$$

$$(b) \quad f(x) = \sqrt{x} \quad [0 \leq x \leq 1];$$

$$(B) \quad f(x) = 2^x \quad [0 \leq x \leq 10].$$

把相应区间等分成 n 份, 求给定函数 $f(x)$ 在相应区间上的积分下和 \underline{S}_n 及积分上和 \overline{S}_n .

解 (a) 把区间 $[-2, 3]$ n 等分, 则每一个子区间的长为 $h = \frac{5}{n}$, 且第 i 个子区间为

$$[-2 + ih, -2 + (i+1)h] \quad (i = 0, 1, \dots, n-1).$$

若令 m_i 及 M_i 分别表示函数 $f(x)$ 在第 i 个子区间上的下确界及上确界, 则因 $f(x) = x^3$ 为增函数, 所以

$$m_i = (-2 + ih)^3,$$

$$M_i = [-2 + (i+1)h]^3 (i = 0, 1, 2, \dots, n-1).$$

于是,

$$\begin{aligned} \underline{S}_n &= \sum_{i=0}^{n-1} m_i \Delta x_i = \sum_{i=0}^{n-1} (-2 + ih)^3 h \\ &= -8nh + 12h^2 \cdot \sum_{i=0}^{n-1} i - 6h^3 \cdot \sum_{i=0}^{n-1} i^2 + h^4 \cdot \sum_{i=0}^{n-1} i^3 \\ &= -40 + \frac{12 \cdot 25n(n-1)}{2n^2} - \frac{125(2n^3 - 3n^2 + n)}{n^3} \\ &\quad + \frac{625(n^4 - 2n^3 + n^2)}{4n^4} \\ &= \frac{65}{4} - \frac{175}{2n} + \frac{125}{4n^2}; \\ \overline{S}_n &= \sum_{i=0}^{n-1} M_i \Delta x_i = \sum_{i=0}^{n-1} [-2 + (i+1)h]^3 \\ &= \frac{65}{4} + \frac{175}{2n} + \frac{125}{4n^2}. \end{aligned}$$

$$(6) \quad h = \frac{1}{n},$$

$$m_i = \sqrt{\frac{i}{n}},$$

$$M_i = \sqrt{\frac{i+1}{n}} (i = 0, 1, 2, \dots, n-1).$$

于是,

$$\underline{S}_n = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \sqrt{\frac{i}{n}} = \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{\frac{i}{n}};$$

$$\overline{S}_n = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \sqrt{\frac{i+1}{n}} = \frac{1}{n} \cdot \sum_{i=1}^n \sqrt{\frac{i}{n}}.$$

$$(B) \quad h = \frac{10}{n},$$

$$m_i = 2^{ih},$$

$$M_i = 2^{(i+1)h} \quad (i = 0, 1, 2, \dots, n-1).$$

于是,

$$\underline{S}_n = \sum_{i=0}^{n-1} h 2^{ih} = \frac{h(2^{nh} - 1)}{2^h - 1} = \frac{10230}{n(2^{\frac{10}{n}} - 1)};$$

$$\begin{aligned} \overline{S}_n &= \sum_{i=0}^{n-1} h 2^{(i+1)h} = \frac{h 2^h (2^{nh} - 1)}{2^h - 1} \\ &= \frac{10230 \cdot 2^{\frac{10}{n}}}{n(2^{\frac{10}{n}} - 1)}. \end{aligned}$$

2183. 分闭区间 $[1, 2]$ 为 n 份,使这分点的横坐标构成一等比级数 $^{*)}$,以求函数 $f(x) = x^4$ 在 $[1, 2]$ 上的积分下和.当 $n \rightarrow \infty$ 时此和的极限等于甚么?

解 设 $\sqrt[n]{2} = q$ 或 $2 = q^n$,分点为

$$1 = q^0 < q^1 < q^2 < \dots < q^n = 2.$$

由于 $f(x) = x^4$ 在 $[1, 2]$ 上为增函数,故积分下和为

$$\begin{aligned} \underline{S}_n &= \sum_{i=0}^{n-1} m_i \Delta x_i = \sum_{i=0}^{n-1} [(q^i)^4 \cdot (q^{i+1} - q^i)] \\ &= (q - 1) \cdot \sum_{i=0}^{n-1} (q^i)^5 = \frac{(q - 1)(q^{5n} - 1)}{q^5 - 1} \\ &= \frac{31 \cdot (\sqrt[n]{2} - 1)}{\sqrt[n]{32} - 1}, \end{aligned}$$

且

$$\begin{aligned} \lim_{n \rightarrow \infty} \underline{S}_n &= 31 \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n]{2} - 1}{\sqrt[n]{32} - 1} \\ &= 31 \cdot \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{16} + \sqrt[n]{8} + \sqrt[n]{4} + \sqrt[n]{2} + 1} \end{aligned}$$

$$= \frac{31}{5}.$$

*) 原题为“使这 n 份的长构成等比级数”, 现根据原题答案予以改正.

2184⁺. 从积分的定义出发, 求

$$\int_0^T (v_0 + gt) dt,$$

其中 v_0 及 g 为常数.

解 $f(t) = v_0 + gt$ 在 $[0, T]$ 上为增函数 ($T > 0$).

$$h = \frac{T}{n},$$

$$m_i = v_0 + igh,$$

$$M_i = v_0 + (i+1)gh \quad (i = 0, 1, 2, \dots, n-1).$$

于是

$$\begin{aligned} \underline{S}_n &= \sum_{i=0}^{n-1} (v_0 + igh) \cdot h = nv_0h + gh^2 \sum_{i=0}^{n-1} i \\ &= v_0T + \frac{gT^2}{n^2} \cdot \frac{n(n-1)}{2} \\ &= v_0T + \frac{gT^2}{2} - \frac{gT^2}{2n}, \end{aligned}$$

$$\overline{S}_n = \sum_{i=0}^{n-1} [v_0 + (i+1)gh]h = v_0T + \frac{gT^2}{2} + \frac{gT^2}{2n}.$$

因为

$$\lim_{n \rightarrow \infty} \underline{S}_n = \lim_{n \rightarrow \infty} \overline{S}_n = v_0T + \frac{gT^2}{2},$$

所以

$$\int_0^T (v_0 + gt) dt = v_0T + \frac{gT^2}{2}.$$

以适当的方法进行积分区间的分段, 把积分看作是

对应的积分和的极限,来计算定积分.

2185. $\int_{-1}^2 x^2 dx.$

解 将区间 $[-1, 2]$ n 等分, 得 $h = \frac{3}{n}$. 取

$$\xi_i = -1 + ih (i = 0, 1, \dots, n-1).$$

作和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} (-1 + ih)^2 h = nh - 2h^2 \sum_{i=0}^{n-1} i + h^3 \sum_{i=0}^{n-1} i^2 \\ &= 3 + \frac{9 - 9n}{2n^2}. \end{aligned}$$

于是

$$\lim_{n \rightarrow \infty} S_n = 3.$$

由于 $f(x) = x^2$ 在 $[-1, 2]$ 上连续, 故积分 $\int_{-1}^2 x^2 dx$ 是存在的, 且它与分法无关, 同时也与点的取法无关. 因此上述和的极限就是所求的积分值(以后如无特殊情况, 不再说明), 即定积分

$$\int_{-1}^2 x^2 dx = 3.$$

2186. $\int_0^1 a^x dx (a > 0).$

解 当 $a \neq 1$ 时, 将区间 $[0, 1]$ n 等分, 得 $h = \frac{1}{n}$.

取

$$\xi_i = ih (i = 0, 1, \dots, n-1).$$

作和

$$S_n = \sum_{i=0}^{n-1} h a^{ih} = \frac{h(a^{nh} - 1)}{a^h - 1} = \frac{a - 1}{n(a^{\frac{1}{n}} - 1)}.$$

于是

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{\frac{a-1}{\frac{1}{a^n} - 1}}{\frac{1}{n}} = \frac{a-1}{\ln a},$$

即

$$\int_0^1 a^x dx = \frac{a-1}{\ln a} \quad (a \neq 1).$$

当 $a = 1$ 时, 积分显然为 1.

2187. $\int_0^{\frac{\pi}{2}} \sin x dx.$

解 将区间 $[0, \frac{\pi}{2}]$ n 等分, 得 $h = \frac{\pi}{2n}$. 取

$$\xi_i = ih \quad (i = 0, 1, \dots, n-1).$$

作和

$$S_n = \sum_{i=0}^{n-1} h \sin ih.$$

由于

$$\sin ih = \frac{1}{2 \sin \frac{h}{2}} \cdot \left[\cos \frac{2i-1}{2} h - \cos \frac{2i+1}{2} h \right],$$

所以

$$\begin{aligned} S_n &= \frac{h}{2 \sin \frac{h}{2}} \sum_{i=0}^{n-1} \left(\cos \frac{2i-1}{2} h - \cos \frac{2i+1}{2} h \right) \\ &= \frac{h}{2 \sin \frac{h}{2}} \left(\cos \frac{h}{2} - \cos \frac{2n-1}{2} h \right). \end{aligned}$$

最后得到

$$\begin{aligned}\lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{\frac{h}{2}}{\sin \frac{h}{2}} \cdot \lim_{n \rightarrow \infty} \left(\cos \frac{\pi}{4n} - \cos \frac{2n-1}{4n} \pi \right) \\ &= 1.\end{aligned}$$

即

$$\int_0^{\frac{\pi}{2}} \sin x dx = 1.$$

2188. $\int_0^x \cos t dt.$

解 将区间 $[0, x]$ n 等分, 得 $h = \frac{x}{n}$. 取

$$\xi_i = ih (i = 0, 1, \dots, n-1).$$

与 2187 题类似, 可得

$$\begin{aligned}& \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} h \cos ih \\ &= \lim_{n \rightarrow \infty} \frac{h}{2 \sin \frac{h}{2}} \cdot \left(\sin \frac{h}{2} + \sin \frac{2n-1}{2} h \right) \\ &= \lim_{n \rightarrow \infty} \frac{\frac{h}{2}}{\sin \frac{h}{2}} \cdot \lim_{n \rightarrow \infty} \left(\sin \frac{x}{2n} + \sin \frac{(2n-1)x}{2n} \right) \\ &= \sin x.\end{aligned}$$

即 $\int_0^x \cos t dt = \sin x.$

2189. $\int_a^b \frac{dx}{x^2} (0 < a < b).$

解 将区间 $[a, b]$ 作 n 等分, 设分点为

$$x_0 = a < x_1 < x_2 < \dots < x_n = b.$$

取 $\xi_i = \sqrt{x_i \cdot x_{i+1}}$ ($i = 0, 1, 2, \dots, n-1$). 显然 $\xi_i \in [x_i, x_{i+1}]$.

作和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} \frac{1}{\sqrt{x_i x_{i+1}}} (x_{i+1} - x_i) \\ &= \sum_{i=0}^{n-1} \left(\frac{1}{x_i} - \frac{1}{x_{i+1}} \right) = \frac{1}{a} - \frac{1}{b}. \end{aligned}$$

于是

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{a} - \frac{1}{b},$$

即

$$\int_a^b \frac{dx}{x^2} = \frac{1}{a} - \frac{1}{b}.$$

2190. $\int_a^b x^m dx \quad (0 < a < b; m \neq -1).$

解 选择诸分点,使它们的横坐标构成一等比级数,即

$$a < aq < aq^2 < \dots < aq^{i-1} < \dots < aq^{n-1} < aq^n = b,$$

其中

$$q = \sqrt[n]{\frac{b}{a}}.$$

取 $\xi_i = aq^i$ ($i = 0, 1, 2, \dots, n-1$). 作和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} (aq^i)^m (aq^{i+1} - aq^i) \\ &= a^{m+1} (q - 1) \cdot \sum_{i=0}^{n-1} q^{(m+1)i} \end{aligned}$$

$$\begin{aligned}
&= a^{m+1}(q-1) \frac{q^{m(m+1)}-1}{q^{m+1}-1} \\
&= (b^{m+1}-a^{m+1}) \cdot \frac{q-1}{q^{m+1}-1}.
\end{aligned}$$

由于 $\lim_{n \rightarrow \infty} q = 1$, 所以

$$\begin{aligned}
\lim_{n \rightarrow \infty} S_n &= (b^{m+1}-a^{m+1}) \cdot \lim_{n \rightarrow \infty} \frac{q-1}{q^{m+1}-1} \\
&= (b^{m+1}-a^{m+1}) \lim_{n \rightarrow \infty} \frac{1}{q^m + q^{m-1} + \cdots + 1} \\
&= \frac{b^{m+1}-a^{m+1}}{m+1},
\end{aligned}$$

即

$$\int_a^b x^m dx = \frac{b^{m+1}-a^{m+1}}{m+1}.$$

2191. $\int_a^b \frac{dx}{x} (0 < a < b).$

解 同 2190 题的分法及取法, 得和

$$\begin{aligned}
S_n &= \sum_{i=0}^{n-1} (aq^i)^{-1} \cdot (aq^{i+1} - aq^i) \\
&= n(q-1) \\
&= n \left[\sqrt[n]{\frac{b}{a}} - 1 \right].
\end{aligned}$$

由于 $\lim_{t \rightarrow 0} \frac{a^t-1}{t} = \ln a (a > 0)$ (可用洛比塔法则), 命

$\alpha = \frac{b}{a}$, 而 $\frac{1}{n}$ 是趋向于 0 的变量, 应用这一极限即得

$$\begin{aligned}
\lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} n \left[\sqrt[n]{\frac{b}{a}} - 1 \right] \\
&= \ln \frac{b}{a},
\end{aligned}$$

而

$$\begin{aligned}
 & \prod_{i=1}^{n-1} \left(1 - 2t \cos \frac{i\pi}{n} + t^2 \right) \\
 &= \prod_{i=1}^{n-1} \left(\sin^2 \frac{i\pi}{n} + \cos^2 \frac{i\pi}{n} - 2t \cos \frac{i\pi}{n} + t^2 \right) \\
 &= \prod_{i=1}^{n-1} \left[\left(t - \cos \frac{i\pi}{n} \right)^2 + \sin^2 \frac{i\pi}{n} \right] \\
 &= \prod_{i=1}^{n-1} \left(t - \cos \frac{i\pi}{n} - j \sin \frac{i\pi}{n} \right) \\
 &\quad \cdot \left(t - \cos \frac{i\pi}{n} + j \sin \frac{i\pi}{n} \right) \\
 &= \prod_{i=1}^{n-1} (t - \epsilon_i)(t - \bar{\epsilon}_i) \\
 &= \frac{t^{2n} - 1}{(t + 1)(t - 1)} \\
 &= \frac{t^{2n} - 1}{t^2 - 1}.
 \end{aligned}$$

即

$$t^{2n} - 1 = (t^2 - 1) \prod_{i=1}^{n-1} \left(1 - 2t \cos \frac{i\pi}{n} + t^2 \right).$$

当 $t = \alpha$ 时, 利用上式就可把 S_n 表成下面的形式

$$S_n = \frac{\pi}{n} \ln \left[\frac{\alpha + 1}{\alpha - 1} (\alpha^{2n} - 1) \right].$$

于是, (a) 当 $|\alpha| < 1$ 时, $\lim_{n \rightarrow \infty} S_n = 0$, 即

$$\int_0^x (1 - 2\alpha \cos x + \alpha^2) dx = 0.$$

(6) 当 $|\alpha| > 1$ 时, 把 S_n 改写成

$$S_n = 2\pi \ln |\alpha| + \frac{\pi}{n} \ln \left[\frac{\alpha + 1}{\alpha - 1} \cdot \frac{\alpha^{2n} - 1}{\alpha^{2n}} \right]$$

后, 由于 $\lim_{n \rightarrow \infty} \frac{\alpha^{2n} - 1}{\alpha^{2n}} = 1$, 从而 $\lim_{n \rightarrow \infty} S_n = 2\pi \ln |\alpha|$,

即

$$\int_0^\pi \ln(1 - 2\alpha \cos x + \alpha^2) dx = 2\pi \ln |\alpha|.$$

2193. 设函数 $f(x)$ 及 $\varphi(x)$ 在 $[a, b]$ 上连续, 证明

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \varphi(\theta_i) \Delta x_i = \int_a^b f(x) \varphi(x) dx.$$

其中 $x_i \leq \xi_i \leq x_{i+1}$, $x_i \leq \theta_i \leq x_{i+1}$ ($i = 0, 1, \dots, n-1$) 及 $\Delta x_i = x_{i+1} - x_i$ ($x_0 = a, x_n = b$).

证 因为 $f(x)$ 及 $\varphi(x)$ 均在 $[a, b]$ 上连续, 所以它们的乘积 $f(x)\varphi(x)$ 也在 $[a, b]$ 上连续. 因此, 积分

$$\int_a^b f(x) \varphi(x) dx = \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \varphi(\xi_i) \Delta x_i \quad (1) \text{ 存在.}$$

由于 $f(x)$ 在 $[a, b]$ 连续, 故有界, 即存在常数 $M > 0$, 使 $|f(x)| \leq M$ ($a \leq x \leq b$); 又由于 $\varphi(x)$ 在 $[a, b]$ 连续, 故一致连续, 因此任给 $\epsilon > 0$, 存在 $\delta > 0$, 使当 $\max |\Delta x_i| < \delta$ 时, 恒有

$$|\varphi(\theta_i) - \varphi(\xi_i)| < \frac{\epsilon}{M(b-a)} \quad (i = 0, 1, \dots, n-1).$$

从而

$$\begin{aligned} & \left| \sum_{i=0}^{n-1} [f(\xi_i) \varphi(\theta_i) - f(\xi_i) \varphi(\xi_i)] \Delta x_i \right| \\ & \leq \sum_{i=0}^{n-1} |f(\xi_i)| \cdot |\varphi(\theta_i) - \varphi(\xi_i)| \cdot |\Delta x_i| \\ & < \sum_{i=0}^{n-1} M \cdot \frac{\epsilon}{M(b-a)} \cdot |\Delta x_i| = \epsilon. \end{aligned}$$

由此可知

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=1}^{n-1} [f(\xi_i)\varphi(\theta_i) - f(\xi_i)\varphi(\xi_i)]\Delta x_i = 0. \quad (2)$$

由(1)式和(2)式,最后得到

$$\int_a^b f(x)\varphi(x)dx = \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i)\varphi(\theta_i)\Delta x_i.$$

2194. 证明不连续的函数:

$$f(x) = \operatorname{sgn}\left(\sin \frac{\pi}{x}\right)$$

于区间 $[0,1]$ 上可积分.

证 首先注意,函数 $f(x) = \operatorname{sgn}\left(\sin \frac{\pi}{x}\right)$ 在 $[0,1]$ 上有界,其不连续点是

$$0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$$

并且, $f(x)$ 在 $[0,1]$ 的任何部分区间上的振幅 $\omega \leq 2$.

任给 $\epsilon > 0$. 由于 $f(x)$ 在 $\left[\frac{\epsilon}{5}, 1\right]$ 上只有有限个间断点,故可积. 因此存在 $\eta > 0$, 使对 $\left[\frac{\epsilon}{5}, 1\right]$ 的任何分法, 只要 $\max |\Delta x'_i| < \eta$, 就有 $\sum_i \omega'_i \Delta x'_i < \frac{\epsilon}{5}$. 显然, 若 $[a, \beta] \subset \left[\frac{\epsilon}{5}, 1\right]$, 则对于 $[a, \beta]$ 的任何分法, 只要 $\max |\Delta x'_i| < \eta$, 也有 $\sum_i \omega'_i \Delta x'_i < \frac{\epsilon}{5}$.

令 $\delta = \min\left\{\frac{\epsilon}{5}, \eta\right\}$. 现设 $0 = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots < x_n = 1$ 是 $[0,1]$ 的满足 $\max |\Delta x_i| < \delta$

的任一分法. 设 $x_{i_0} \leq \frac{\varepsilon}{5} < x_{i_0+1}$.

由上述, 有 $\sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \frac{\varepsilon}{5}$. 又, 显然

$$\sum_{i=0}^{i_0} \omega_i \Delta x_i \leq 2 \sum_{i=0}^{i_0} \Delta x_i < 2 \cdot \frac{2\varepsilon}{5} = \frac{4\varepsilon}{5}.$$

故

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i = \sum_{i=0}^{i_0} \omega_i \Delta x_i + \sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \varepsilon.$$

由此可知

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0.$$

于是, $f(x)$ 在 $[0, 1]$ 可积.

2195. 证明黎曼函数

$$\varphi(x) = \begin{cases} 0, & \text{若 } x \text{ 为无理数,} \\ \frac{1}{n}, & \text{若 } x = \frac{m}{n}, \end{cases}$$

(式中 m 及 $n (n \geq 1)$ 为互质的整数) 在任何有穷的区间上可积分.

证 为简单起见, 我们只考虑闭区间 $[0, 1]$ (对于一般的有限闭区间 $[a, b]$, 可类似地讨论之).

命 $\lambda > 0$ 将区间 $[0, 1]$ 分成长度 $\Delta x_i < \lambda$ 的若干部分区间, 取任意的自然数 N , 将所有的部分区间分成两类:

把包含分母 $n \leq N$ 的数 $\frac{m}{n}$ 的区间列入第一类, 而把不包含上述数的那些区间列入第二类. 对于第一类, 由于满足条件 $n \leq N$ 的数 $\frac{m}{n}$ 只有有限个, 个数记为 $k = k_N$,

所以第一类区间的个数就不大于 $2k$, 而它们长度的总和不超出 $2k\lambda$; 对于第二类, 由于在这些区间内除含有无理数外, 仅能含 $n > N$ 的有理数 $\frac{m}{n}$, 而在这种有理点上, $\varphi\left(\frac{m}{n}\right) = \frac{1}{n} < \frac{1}{N}$, 所以, 振幅 ω_i 小于 $\frac{1}{N}$.

这样一来, 和数 $\sum_{i=0}^{n-1} \omega_i \Delta x_i$ 就分成两部分, 分别估计它们的值, 即得

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i < 2k_N \lambda + \frac{1}{N}.$$

对于任意给定的 $\epsilon > 0$, 取定一个 $N > \frac{2}{\epsilon}$, 然后取 $\delta = \frac{\epsilon}{4k_N}$, 于是, 当 $\lambda < \delta$ 时, 必有

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i < \epsilon,$$

故

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0.$$

所以函数 $\varphi(x)$ 在 $[0, 1]$ 上可积分.

2196. 证明函数

$$\text{当 } x \neq 0, f(x) = \frac{1}{x} - \left[\frac{1}{x} \right] \text{ 及 } f(0) = 0,$$

于闭区间 $[0, 1]$ 上可积分.

证 首先注意, 函数 $f(x)$ 在 $[0, 1]$ 上有界, 其不连续点是

$$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$$

并且, $f(x)$ 在 $[0, 1]$ 的任何部分区间上的振幅 $\omega \leq 1$.

任给 $\varepsilon > 0$. 由于 $f(x)$ 在 $\left[\frac{\varepsilon}{3}, 1\right]$ 上只有有限个间断点, 故可积. 因此, 存在 $\eta > 0$, 使对 $\left[\frac{\varepsilon}{3}, 1\right]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 就有 $\sum_i \omega_i \Delta x_i < \frac{\varepsilon}{3}$. 显然, 若 $[\alpha, \beta] \subset \left[\frac{\varepsilon}{3}, 1\right]$, 则对于 $[\alpha, \beta]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 也有 $\sum_i \omega_i \Delta x_i < \frac{\varepsilon}{3}$.

令 $\delta = \min \left\{ \frac{\varepsilon}{3}, \eta \right\}$. 现设 $0 = x_0 < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_n = 1$ 是 $[0, 1]$ 的满足 $\max |\Delta x_i| < \delta$ 的任一分法. 设 $x_{i_0} \leq \frac{\varepsilon}{3} < x_{i_0+1}$. 由上述, 有

$$\sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \frac{\varepsilon}{3}.$$

又, 显然 $\sum_{i=0}^{i_0} \omega_i \Delta x_i \leq \sum_{i=0}^{i_0} \Delta x_i < \frac{2\varepsilon}{3}$. 故

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i = \sum_{i=0}^{i_0} \omega_i \Delta x_i + \sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \varepsilon.$$

于是

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0.$$

由此可知, $f(x)$ 在 $[0, 1]$ 上可积.

2197. 证明迪里黑里函数

$$\chi(x) = \begin{cases} 0, & \text{若 } x \text{ 为无理数,} \\ 1, & \text{若 } x \text{ 为有理数,} \end{cases}$$

于任意区间上不可积分.

证 在任意区间 $[a, b]$ 的任何部分区间上均有

$$\omega_i = 1,$$

所以 $\sum_{i=0}^{n-1} \omega_i \Delta x_i = b - a$, 它不趋于零. 因此函数 $\chi(x)$ 在 $[a, b]$ 上不可积分.

2198. 设函数 $f(x)$ 于 $[a, b]$ 上可积分, 且

$$f_n(x) = \sup f(x) \quad \text{当 } x_i < x \leq x_{i+1},$$

其中 $x_i = a + \frac{i}{n}(b-a) \left(\begin{matrix} i = 0, 1, \dots, n-1; \\ n = 1, 2, \dots \end{matrix} \right).$

证明 $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$

证 $f_n(x)$ 是不超过 $n+1$ 个间断点的阶梯函数, 因此 $f_n(x)$ 在 $[a, b]$ 上可积分, 于是

$$\begin{aligned} & \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| \\ & \leq \int_a^b |f_n(x) - f(x)| dx \\ & = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f_n(x) - f(x)| dx \\ & \leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \omega_i dx = \sum_{i=0}^{n-1} \omega_i \Delta x_i \rightarrow 0 \\ & \left(\text{当 } \max |\Delta x_i| = \frac{b-a}{n} \rightarrow 0 \text{ 时} \right), \end{aligned}$$

即

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

2199. 证明: 若函数 $f(x)$ 于 $[a, b]$ 上可积分, 则有连续函数 $\varphi_n(x) (n = 1, 2, \dots)$ 的叙列, 使

$$\int_a^c f(x) dx = \lim_{n \rightarrow \infty} \int_a^c \varphi_n(x) dx, \text{ 当 } a \leq c \leq b.$$

证 将区间 $[a, b]$ 作 n 等分, 设分点为

$$a = x_0^{(n)} < x_1^{(n)} < \dots < x_{n-1}^{(n)} < x_n^{(n)} = b,$$

即 $x_i^{(n)} = a + \frac{i}{n}(b-a), i = 0, 1, \dots, n.$

在 $\Delta x_i^{(n)} = [x_{i-1}^{(n)}, x_i^{(n)}]$ 上令 $\varphi_n(x)$ 为过点 $[x_{i-1}^{(n)}, f(x_{i-1}^{(n)})]$ 及 $[x_i^{(n)}, f(x_i^{(n)})]$ 的直线, 即当 $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$ 时, 令

$$\varphi_n(x) = f(x_{i-1}^{(n)}) + \frac{x - x_{i-1}^{(n)}}{x_i^{(n)} - x_{i-1}^{(n)}} [f(x_i^{(n)}) - f(x_{i-1}^{(n)})],$$

则 $\varphi_n(x)$ 是 $[a, b]$ 上的连续函数, 因此, 它是可积分的.

若令 $m_i^{(n)}, M_i^{(n)}$ 及 $\omega_i^{(n)}$ 分别表示函数 $f(x)$ 在 $[x_{i-1}^{(n)}, x_i^{(n)}]$ 上的下确界, 上确界及振幅, 则当 $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$ 时,

$$m_i^{(n)} \leq \varphi_n(x) \leq M_i^{(n)}, m_i^{(n)} \leq f(x) \leq M_i^{(n)},$$

从而

$$|\varphi_n(x) - f(x)| \leq \omega_i^{(n)}.$$

于是, 当 $a \leq c \leq b$ 时,

$$\begin{aligned} & \left| \int_a^c f(x) dx - \int_a^c \varphi_n(x) dx \right| \\ & \leq \int_a^c |f(x) - \varphi_n(x)| dx \\ & \leq \int_a^b |f(x) - \varphi_n(x)| dx \end{aligned}$$

$$= \sum_{i=1}^n \int_{x_{i-1}^{(n)}}^{x_i^{(n)}} |f(x) - \varphi_n(x)| dx$$

$$\leq \sum_{i=1}^n \omega_i^{(n)} \Delta x_i^{(n)}.$$

由于 $f(x)$ 在 $[a, b]$ 上可积分, 因此,

当 $\max |\Delta x_i^{(n)}| = \frac{b-a}{n} \rightarrow 0$ 时, 必有

$$\sum_{i=1}^n \omega_i^{(n)} \Delta x_i^{(n)} \rightarrow 0.$$

由此可知

$$\int_a^c f(x) dx = \lim_{n \rightarrow \infty} \int_a^c \varphi_n(x) dx \quad (a \leq c \leq b).$$

2200. 证明: 若有界的函数 $f(x)$ 于闭区间 $[a, b]$ 上可积分, 则其绝对值 $|f(x)|$ 于 $[a, b]$ 上也可积分, 并且

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

证 对于区间 $[x_i, x_{i+1}]$ 上任意两点 x' 及 x'' , 总有

$$||f(x')| - |f(x'')|| \leq |f(x') - f(x'')|,$$

所以函数 $|f(x)|$ 在 $[x_i, x_{i+1}]$ 上的振幅 ω_i^* 不超过 $f(x)$ 在此区间上的振幅 ω_i , 因而

$$\sum_{i=0}^{n-1} \omega_i^* \Delta x_i \leq \sum_{i=0}^{n-1} \omega_i \Delta x_i \rightarrow 0,$$

即 $|f(x)|$ 在 $[a, b]$ 上可积分.

其次, 因为

$$-|f(x)| \leq f(x) \leq |f(x)|,$$

所以

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx,$$

$< \delta$, 就有 $\sum_{i=0}^{n-1} \omega_i(f) \Delta x_i < \frac{\eta \epsilon}{2\Omega}$. ($\omega_i(f)$ 表 $f(x)$ 在 $[x_i, x_{i+1}]$ 上的振幅).

下证对 $[a, b]$ 的任何分法, 只要 $\max |\Delta x_i| < \delta$, 就有

$$\sum_{i=0}^{n-1} \omega_i(\varphi(f)) \Delta x_i < \epsilon.$$

事实上, 将诸区间 $[x_i, x_{i+1}]$ 分成两组, 第一组是满足 $\omega_i(f) < \eta$ 的 (其下标以 “ i' ” 记之), 第二组是满足 $\omega_i(f) \geq \eta$ 的 (下标以 “ i'' ” 记之).

于是,

$$\begin{aligned} & \sum_{i=0}^{n-1} \omega_i(\varphi(f)) \Delta x_i \\ &= \sum_{i'} \omega_{i'}(\varphi(f)) \Delta x_{i'} + \sum_{i''} \omega_{i''}(\varphi(f)) \Delta x_{i''} \\ &< \frac{\epsilon}{2(b-a)} \sum_{i'} \Delta x_{i'} + \Omega \sum_{i''} \Delta x_{i''}, \end{aligned}$$

但

$$\begin{aligned} & \frac{\eta \epsilon}{2\Omega} > \sum_{i=0}^{n-1} \omega_i(f) \Delta x_i \\ &= \sum_{i'} \omega_{i'}(f) \Delta x_{i'} + \sum_{i''} \omega_{i''}(f) \Delta x_{i''} \\ &\geq \sum_{i''} \omega_{i''}(f) \Delta x_{i''} \geq \eta \sum_{i''} \Delta x_{i''}, \end{aligned}$$

于是

$$\sum_{i=0}^{n-1} \omega_i(\varphi(f)) \Delta x_i$$

$$< \frac{\varepsilon}{2(b-a)} \cdot (b-a) + \Omega \cdot \frac{\varepsilon}{2\Omega} = \varepsilon.$$

由此可知, $\varphi[f(x)]$ 在 $[a, b]$ 上可积.

2203. 若函数 $f(x)$ 及 $\varphi(x)$ 可积分, 则函数 $f[\varphi(x)]$ 是否也必定可积分?

解 未必. 例如函数

$$f(x) = \begin{cases} 0, & \text{若 } x = 0, \\ 1 & \text{若 } x \neq 0, \end{cases}$$

及 $\varphi(x)$ 为黎曼函数(参阅 2195 题).

它们在任何有穷的区间上均可积(前者不连续点仅为原点一个, 且是有界函数, 因而是可积分的).

但 $f[\varphi(x)] = \chi(x)$, 利用 2197 题的结果得知它在任何有穷的区间上不可积分.

2204. 设函数 $f(x)$ 于闭区间 $[A, B]$ 上可积分, 证明函数 $f(x)$ 有积分的连续性, 即是说

$$\lim_{h \rightarrow 0} \int_a^{a+h} |f(x+h) - f(x)| dx = 0,$$

式中 $[a, b] \subset [A, B]$.

证 方法一:

不妨设 $A < a, b < B$. 由于 $f(x)$ 在 $[A, B]$ 可积, 故任给 $\varepsilon > 0$, 存在 $\eta > 0$, 使对 $[A, B]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 就恒有

$$\sum_i \omega_i \Delta x_i < \varepsilon;$$

显然, 对 $[A, B]$ 的任一子区间 $[A', B']$ 的任何分法, 只要 $\max |\Delta x_{i'}| < \eta$, 也有

$$\sum_{i'} \omega_{i'} \Delta x_{i'} < \varepsilon. \quad (1)$$

今设 $0 < h < \delta = \min\left\{\frac{\eta}{2}, \frac{B-b}{3}\right\}$, 则对于 h , 存在正整数 $n = n(h)$, 使有
 $a + (2n-2)h < b \leq a + 2nh < a + (2n+1)h < B$.
 用 ω_i 表 $f(x)$ 在 $[a + ih, a + (i+2)h]$ 上的振幅, 则

$$\begin{aligned} & \int_a^b |f(x+h) - f(x)| dx \\ & \leq \int_a^{a+2nh} |f(x+h) - f(x)| dx \\ & = \sum_{i=0}^{2n-1} \int_{a+ih}^{a+(i+1)h} |f(x+h) - f(x)| dx \leq \sum_{i=0}^{2n-1} \omega_i h \\ & = \frac{1}{2} \sum_{i=0}^{n-1} \omega_{2i} 2h + \frac{1}{2} \sum_{i=0}^{n-1} \omega_{2i+1} 2h. \end{aligned}$$

显然, $\sum_{i=0}^{n-1} \omega_{2i} \cdot 2h$ 是对于区间 $[a, a + 2nh]$ 的分法 $a < a + 2h < a + 4h < \dots < a + 2nh$ 所作的(1)式中的和, 而 $\sum_{i=0}^{n-1} \omega_{2i+1} 2h$ 是对于区间 $[a + h, a + (2n+1)h]$ 的分法

$a + h < a + 3h < a + 5h < \dots < a + (2n+1)h$ 所作的(1)式中的和. 故

$$\sum_{i=0}^{n-1} \omega_{2i} 2h < \epsilon, \quad \sum_{i=0}^{n-1} \omega_{2i+1} 2h < \epsilon.$$

从而

$$\int_a^b |f(x+h) - f(x)| dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

由此可知

$$\lim_{h \rightarrow 0+} \int_b^a |f(x+h) - f(x)| dx = 0.$$

同理可证

$$\lim_{h \rightarrow 0-} \int_a^b |f(x+h) - f(x)| dx = 0.$$

于是,得

$$\lim_{h \rightarrow 0} \int_a^b |f(x+h) - f(x)| dx = 0.$$

方法二:

由 2199 题的结果可知:对于任意给定的 $\epsilon > 0$, 由于 $f(x)$ 在 $[A, B]$ 上可积, 故存在 $[A, B]$ 上的连续函数 $\varphi(x)$, 使

$$\int_A^B |f(x) - \varphi(x)| dx < \frac{\epsilon}{4}.$$

由于 $\varphi(x)$ 在 $[A, B]$ 上一致连续, 故存在 $\delta > 0$, 使当 $|x' - x''| < \delta$ ($x' \in [A, B], x'' \in [A, B]$) 时, 恒有

$$|\varphi(x') - \varphi(x'')| < \frac{\epsilon}{2(b-a)}.$$

于是, 当 $|h| < \delta$ 时,

$$\begin{aligned} & \int_a^b |f(x+h) - f(x)| dx \\ & \leq \int_a^b |f(x+h) - \varphi(x+h)| dx \\ & \quad + \int_a^b |\varphi(x+h) - \varphi(x)| dx \\ & \quad + \int_a^b |f(x) - \varphi(x)| dx \\ & \leq 2 \int_A^B |f(x) - \varphi(x)| dx \\ & \quad + \int_a^b |\varphi(x+h) - \varphi(x)| dx \end{aligned}$$

$$< 2 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{2(b-a)}(b-a) = \epsilon.$$

故

$$\lim_{h \rightarrow 0} \int_a^b |f(x+h) - f(x)| dx = 0.$$

2205. 设函数 $f(x)$ 于闭区间 $[a, b]$ 上可积分, 证明等式

$$\int_a^b f^2(x) dx = 0$$

当而且仅当对属于闭区间 $[a, b]$ 内函数 $f(x)$ 连续的一切点有 $f(x) = 0$ 时方成立.

证 先证必要性:

采用反证法. 设 $f(x)$ 在点 x_0 连续, 但 $f(x_0) \neq 0$, 则存在 $\delta > 0$, $[x_0 - \delta, x_0 + \delta] \subset [a, b]$, 使当 $|x - x_0| \leq \delta$ 时

$$|f(x)| > \frac{|f(x_0)|}{2}.$$

从而

$$\begin{aligned} \int_a^b f^2(x) dx &\geq \int_{x_0-\delta}^{x_0+\delta} f^2(x) dx > \frac{f^2(x_0)}{4} \cdot 2\delta \\ &= \frac{\delta \cdot f^2(x_0)}{2} > 0. \end{aligned}$$

这与假设 $\int_a^b f^2(x) dx = 0$ 矛盾.

再证充分性:

也即要证: $f(x)$ 在 $[a, b]$ 上可积条件下, 假设 $f(x)$ 在一切连续点 x_0 上均有 $f(x_0) = 0$, 则必有

$$\int_a^b f^2(x) dx = 0.$$

证明分两个部分. 第一, 首先要指出当 $f(x)$ 在

$[a, b]$ 上可积时, $f(x)$ 的连续点在 $[a, b]$ 中必定是稠密的. 此处所谓“稠密”性是指: 对于任意区间 $[\alpha, \beta] \subset [a, b]$ 总存在一点 $x_0 \in [\alpha, \beta]$, 使 $f(x)$ 在 x_0 连续. 第二, 利用假设, 并借助于稠密性, 可证得充分性. 现在先证第二部分, 如下: 由 $f(x)$ 在 $[a, b]$ 上的全体连续点 X 的稠密性以及当 $x_0 \in X$ 时有 $f(x_0) = 0$ 的假设. 便知, 对于区间 $[a, b]$ 的任一分法, 均可适当地取 $x_i \leq$

$\xi_i \leq x_{i+1}$, 使 $f(\xi_i) = 0$. 从而积分和 $\sum_{i=0}^{n-1} f^2(\xi_i) \Delta x_i = 0$.

由此, 再注意到 $f^2(x)$ 在 $[a, b]$ 的可积性, 便有

$$\int_a^b f^2(x) dx = \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f^2(\xi_i) \Delta x_i = 0.$$

如今再补证第一部分: 应当首先指明, 若 $f(x)$ 在 $[\alpha, \beta]$ 上可积, 则对任给的 $\varepsilon > 0$, 总存在 $[\alpha, \beta]$ 的子区间 $[\alpha', \beta']$ 使得振幅

$$\omega(\alpha', \beta') < \varepsilon.$$

事实上, 如果上述结论不成立, 则存在一个 $\varepsilon_0 > 0$, 使对于 $[\alpha, \beta]$ 的任意分法, 有

$$\sum_i \omega_i \Delta x_i \geq \varepsilon_0 \sum_i \Delta x = \varepsilon_0 (\beta - \alpha) > 0,$$

这与 $f(x)$ 在 $[\alpha, \beta]$ 可积矛盾, 因此, 结论为真.

今取 $[\alpha, \beta]$ 为 $[a_1, b_1]$. 由于 $f(x)$ 在 $\left[a_1 + \frac{b_1 - a_1}{4}, b_1 - \frac{b_1 - a_1}{4}\right]$ 上可积, 故存在区间 $[a_2, b_2] \subset \left[a_1 + \frac{b_1 - a_1}{4}, b_1 - \frac{b_1 - a_1}{4}\right] \subset [a_1, b_1]$, 使

$$\omega(a_2, b_2] < \frac{1}{2},$$

同样, 存在区间 $[a_3, b_3] \subset \left[a_2 + \frac{b_2 - a_2}{4}, b_2 - \frac{b_2 - a_2}{4} \right] \subset [a_2, b_2]$, 使

$$\omega[a_3, b_3] < \frac{1}{3}.$$

这样继续下去, 得一串闭区间 $[a_n, b_n] (n = 1, 2, 3, \dots)$, 满足

$a = a_1 < a_2 < \dots < a_n < \dots < b_n < \dots < b_2 < b_1 = \beta$, 并且 $b_n - a_n \leq \frac{\beta - a}{2^{n-1}} \rightarrow 0, \omega[a_n, b_n] < \frac{1}{n} (n = 1, 2, 3, \dots)$.

由区间套定理, 诸 $[a_n, b_n]$ 具有唯一的公共点 c . 显然 $a_n < c < b_n (n = 1, 2, 3, \dots)$. 下证 $f(x)$ 在点 c 连续.

任给 $\varepsilon > 0$, 取正整数 n_0 使 $n_0 > \frac{1}{\varepsilon}$. 再取 $\delta > 0$ 使 $(c - \delta, c + \delta) \subset [a_{n_0}, b_{n_0}]$. 于是, 当 $|x - c| < \delta$ 时, 必有

$$|f(x) - f(c)| \leq \omega[a_{n_0}, b_{n_0}] < \frac{1}{n_0} < \varepsilon.$$

故 $f(x)$ 在点 $x = c$ 连续. 到此, 充分性证毕.

§ 2. 利用不定积分计算定积分的方法

1° 牛顿—莱布尼兹公式 若函数 $f(x)$ 于闭区间 $[a, b]$ 上有定义而且连续, $F(x)$ 为它的原函数 (即 $F'(x) = f(x)$), 则

$$\int_a^b f(x)dx = F(b) - F(a) = F(x) \Big|_a^b.$$

定积分 $\int_a^b f(x)dx$ 的几何意义表示由曲线 $y=f(x)$, OX 轴及垂直于 OX 轴的二直线 $x=a$ 和 $x=b$ 四者所围成的代数面积 S (图 4.1).

2° 部分积分法 若函数 $f(x)$ 和 $g(x)$ 于闭区间 $[a, b]$ 上连续并有连续导数 $f'(x)$ 和 $g'(x)$, 则

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x)dx.$$

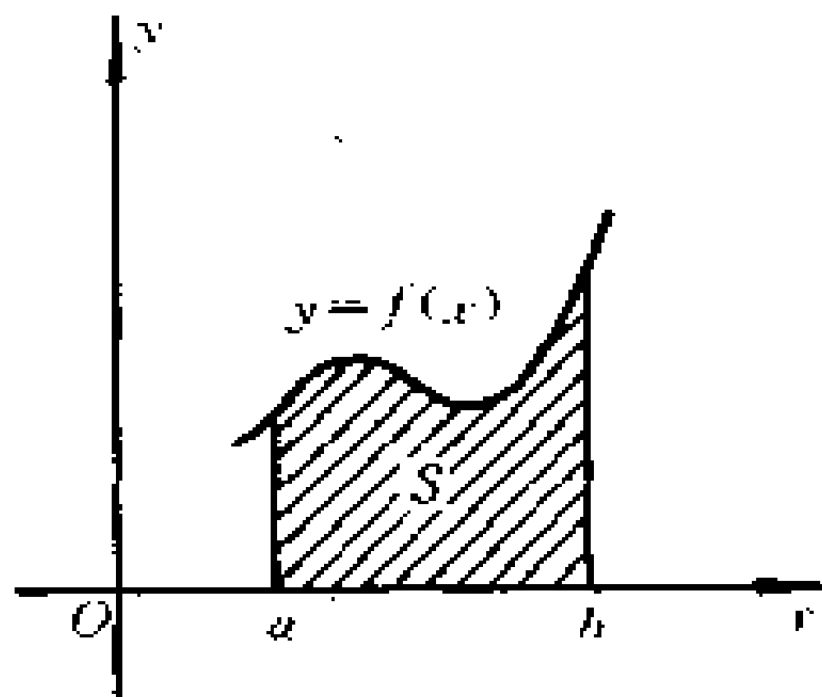


图 4.1

3° 变数代换 若: (1) 函数 $f(x)$ 于闭区间 $[a, b]$ 内连续, (2) 函数 $\varphi(t)$ 及其导数 $\varphi'(t)$ 皆于闭区间 $[\alpha, \beta]$ 上连续, 其中 $a = \varphi(\alpha)$, $b = \varphi(\beta)$; (3) 复合函数 $f(\varphi(t))$ 于闭区间 $[\alpha, \beta]$ 上有定义并连续, 则

$$\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

利用牛顿—莱布尼兹公式, 求下列定积分并绘出对应的曲边图形面积:

2206. $\int_{-1}^8 \sqrt[3]{x} dx$

解 $\int_{-1}^8 \sqrt[3]{x} dx = \frac{3}{4} x^{\frac{4}{3}} \Big|_{-1}^8$
 $= 11 \frac{1}{4}$ (图 4.2).

2207. $\int_0^{\pi} \sin x dx$

* 本节个别题是收敛的广义积分, 仍按此公式计算. —— 题解编者注.

解 $\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = 2$ (图 4.3).

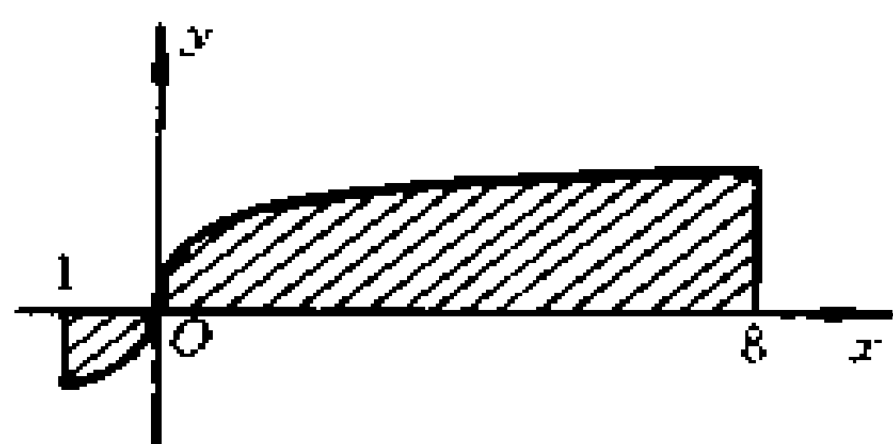


图 4.2

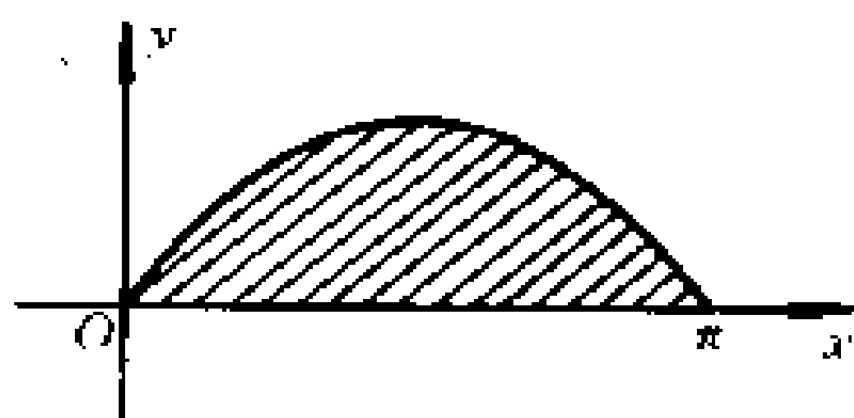


图 4.3

2208. $\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$

解 $\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \operatorname{arctg} x \Big|_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{\pi}{6}$ (图 4.4).

2209. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

解 $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3}$ (图 4.5).

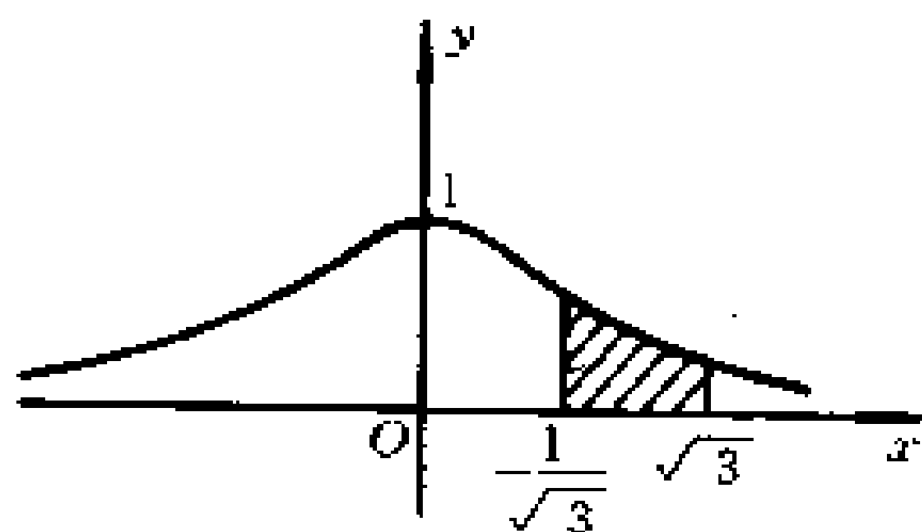


图 4.4

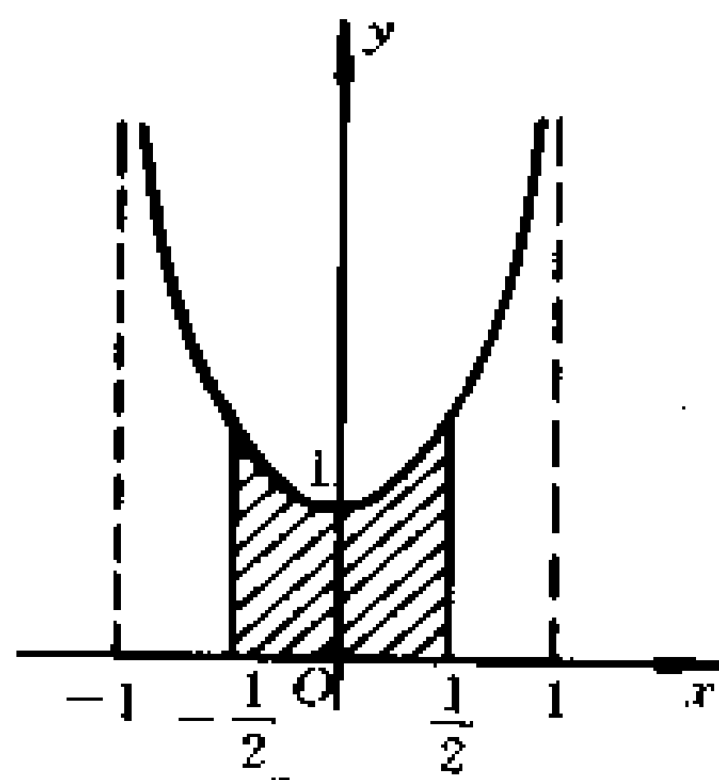


图 4.5

$$= \frac{\pi}{2\sin\alpha} \quad (\text{图 4.8}).$$

注 以下图形从略.

2213. $\int_0^{2\pi} \frac{dx}{1 + \epsilon \cos x}$
 $(0 \leq \epsilon < 1).$

解

$$\begin{aligned} & \int_0^{2\pi} \frac{dx}{1 + \epsilon \cos x} \\ &= \int_0^{\pi} \frac{dx}{1 + \epsilon \cos x} \\ & \quad + \int_{\pi}^{2\pi} \frac{dx}{1 + \epsilon \cos x} \\ &= \int_0^{\pi} \frac{dx}{1 + \epsilon \cos x} + \int_{\pi}^0 \frac{d(2\pi - x)}{1 + \epsilon \cos(2\pi - x)} \\ &= 2 \int_0^{\pi} \frac{dx}{1 + \epsilon \cos x} \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \epsilon \cos x} + 2 \int_{\frac{\pi}{2}}^{\pi} \frac{dx}{1 + \epsilon \cos x} \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \epsilon \cos x} + 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 - \epsilon \cos x} \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{dx}{1 - \epsilon^2 \cos^2 x} \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{dx}{(1 - \epsilon^2) \cos^2 x + \sin^2 x} \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{d \operatorname{tg} x}{\operatorname{tg}^2 x + (1 - \epsilon^2)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{4}{\sqrt{1 - \epsilon^2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} x}{\sqrt{1 - \epsilon^2}} \right) \\ &= \frac{4}{\sqrt{1 - \epsilon^2}} \cdot \frac{\pi}{2} = \frac{2\pi}{\sqrt{1 - \epsilon^2}}. \end{aligned}$$

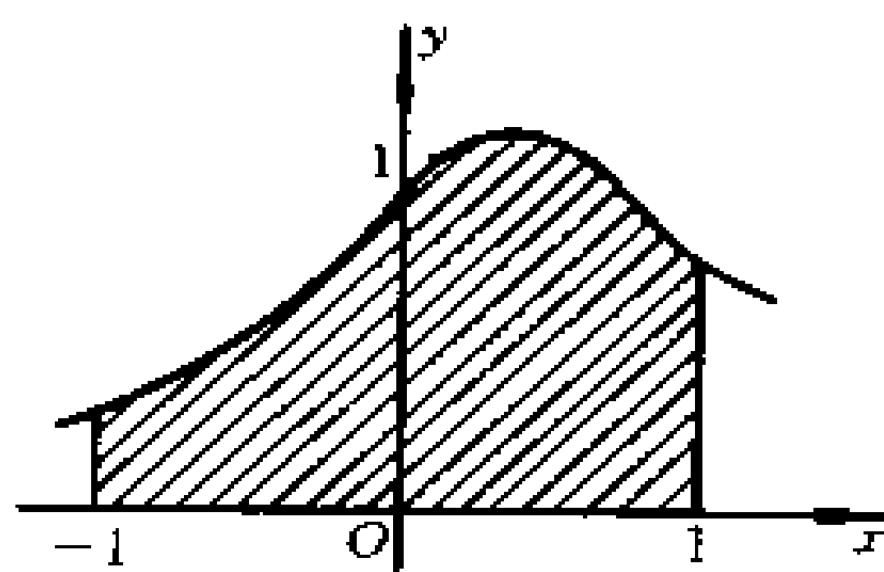


图 4.8

$$2214. \int_{-1}^1 \frac{dx}{\sqrt{(1-2ax+a^2)(1-2bx+b^2)}} \quad (|a| < 1, \\ |b| < 1, ab > 0).$$

解 在公式

$$\int \frac{dx}{\sqrt{Ax^2+Bx+C}} \\ = \frac{1}{\sqrt{A}} \ln \left| Ax + \frac{B}{2} + \sqrt{A} \cdot \sqrt{Ax^2+Bx+C} \right| \\ + C^*)$$

中, 设

$$Ax^2+Bx+C = (1-2ax+a^2)(1-2bx+b^2),$$

两端求导数得

$$Ax + \frac{B}{2} = -b(1-2ax+a^2) - a(1-2bx+b^2).$$

由此推得, 当 $x=1$ 时, 在对数符号下的表达式的值为

$$-a(1-b)^2 - b(1-a)^2 + 2\sqrt{ab}(1-a)(1-b) \\ = -(\sqrt{a}-\sqrt{b})^2(1+\sqrt{ab})^2,$$

而当 $x=-1$ 时, 得到值 $-(\sqrt{a}-\sqrt{b})^2(1-\sqrt{ab})^2$.

于是,

$$\int_{-1}^1 \frac{dx}{\sqrt{(1-2ax+a^2)(1-2bx+b^2)}} \\ = \frac{1}{\sqrt{ab}} \ln \frac{1+\sqrt{ab}}{1-\sqrt{ab}}.$$

*) 利用 1850 题的结果.

$$2215. \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad (ab \neq 0).$$

解
$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$$

$$= \frac{1}{|ab|} \arctan \left(\frac{|a| \operatorname{tg} x}{|b|} \right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2|ab|}.$$

 *) 利用 2030 题的结果.

2216. 设

(a) $\int_{-1}^1 \frac{dx}{x^2}$; (6) $\int_0^{2\pi} \frac{\sec^2 x dx}{2 + \operatorname{tg}^2 x}$;

(B) $\int_{-1}^1 \frac{d}{dx} \left(\arctan \frac{1}{x} \right) dx.$

说明为甚么运用牛顿—莱布尼兹公式会得到不正确的结果.

解 (a) 若应用公式得

$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_{-1}^1 = -2 < 0.$$

这是不正确的. 事实上, 由于函数 $f(x) = \frac{1}{x^2} > 0$, 所以当积分存在时, 其值必大于零. 原因在于该函数在区间 $[-1, 1]$ 上有第二类间断点 $x = 0$. 因而不能运用公式.

(6) 若应用公式得

$$\begin{aligned} & \int_0^{2\pi} \frac{\sec^2 x dx}{2 + \operatorname{tg}^2 x} \\ &= \frac{1}{\sqrt{2}} \arctan \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right) \Big|_0^{2\pi} = 0. \end{aligned}$$

但 $\frac{\sec^2 x}{2 + \operatorname{tg}^2 x} > 0$, 故积分若存在, 必为正. 原因在于原函数在 $[0, 2\pi]$ 上 $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ 为第一类不连续点,

故不能直接运用公式.

(B) 若应用公式得

$$\begin{aligned} & \int_{-1}^1 \frac{d}{dx} \left(\arctan \frac{1}{x} \right) dx \\ &= \arctan \frac{1}{x} \Big|_{-1}^1 = \frac{\pi}{2} > 0. \end{aligned}$$

这是不正确的, 因为 $\frac{d}{dx} \left(\arctan \frac{1}{x} \right) = -\frac{1}{1+x^2} < 0$.

所以, 积分值必为负. 原因在于原函数 $\arctan \frac{1}{x}$ 在 $x=0$ 为第一类不连续点, 故不能直接运用公式.

2217. 求 $\int_{-1}^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx$.

解 我们有

$$\begin{aligned} & \int_{-1}^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx \\ &= \int_{-1}^0 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx + \int_0^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx \\ &= \frac{1}{1+2^{\frac{1}{x}}} \Big|_{-1}^0 + \frac{1}{1+2^{\frac{1}{x}}} \Big|_0^1 = \frac{2}{3}. \end{aligned}$$

注意, 被积函数 $\frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right)$ 显然在 $x=0$ 间断, 但易

知 $\lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) = 0$, 故 $x=0$ 是可去间断点. 若我们补充定义被积函数在 $x=0$ 时的值为 0, 则被积函数在整个 $(-1, 1)$ 上都是连续的, 从而积分

$\int_{-1}^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx$ 存在. 以后, 凡是被积函数有可去

间断点的情形, 我们都按此法处理, 理解为连续函数的

积分, 另外,

$$\int_{-1}^0 \frac{d}{dx} \left(\frac{1}{1 + 2^{\frac{1}{x}}} \right) dx = \frac{1}{1 + 2^{\frac{1}{x}}} \Big|_{-1}^0 = \frac{1}{3}$$

应理解为

$$\begin{aligned} \int_{-1}^0 \frac{d}{dx} \left(\frac{1}{1 + 2^{\frac{1}{x}}} \right) dx &= \lim_{\eta \rightarrow 0+} \int_{-1}^{-\eta} \frac{d}{dx} \left(\frac{1}{1 + 2^{\frac{1}{x}}} \right) dx \\ &= \lim_{\eta \rightarrow 0+} \frac{1}{1 + 2^{\frac{1}{x}}} \Big|_{-1}^{-\eta} = \frac{1}{3}. \end{aligned}$$

以后, 凡是定积分存在而原函数有间断点的情况, 都按此理解, 省去取极限的式子, 但应理解为取极限的结果.

2218. 求 $\int_0^{100\pi} \sqrt{1 - \cos 2x} dx$

解
$$\begin{aligned} &\int_0^{100\pi} \sqrt{1 - \cos 2x} dx \\ &= \sum_{k=1}^{100} \sqrt{2} \int_{(k-1)\pi}^{k\pi} \sqrt{\sin^2 x} dx \\ &= \sum_{k=1}^{100} \int_0^{\pi} \sqrt{\sin^2 x} dx \\ &= 100 \sqrt{2} \int_0^{\pi} \sin x dx = 200 \sqrt{2}. \end{aligned}$$

利用定积分求下列和的极限值:

2219. $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2} \right).$

解 这是和的极限, 该极限即为函数 $f(x) = x$ 在区间 $[0, 1]$ 上的定积分. 事实上, 函数 $f(x) = x$ 在 $[0, 1]$ 上是连续的, 因而可积分. 这样便可将 $[0, 1]$ n 等份, 并取 ξ_i 为小区间的左端点, 这样作出的和的极限就是题中所要求的极限. 于是,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \int_0^1 x dx = \frac{1}{2}. \end{aligned}$$

以下各题不再说明.

$$2220. \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right).$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n+i} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{1 + \frac{i}{n}} \cdot \frac{1}{n} \\ &= \int_0^1 \frac{1}{1+x} dx = \ln 2. \end{aligned}$$

$$2221. \lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} \right).$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n}{n^2+i^2} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{1 + \left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} \\ &= \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}. \end{aligned}$$

$$2222. \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \cdots + \sin \frac{(n-1)\pi}{n} \right).$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \frac{1}{n} \sin \frac{i\pi}{n} &= \int_0^1 \sin \pi x dx \\ &= -\frac{1}{\pi} \cos \pi x \Big|_0^1 = \frac{2}{\pi}. \end{aligned}$$

$$2223. \lim_{n \rightarrow \infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} \quad (p > 0).$$

$$\text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n} \right)^p \cdot \frac{1}{n} = \int_0^1 \frac{1}{x^p} dx = \frac{1}{p+1}.$$

$$2224. \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{2}{n}} + \cdots + \sqrt{1 + \frac{n}{n}} \right].$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \sqrt{1 + \frac{i}{n}} &= \int_0^1 \sqrt{1+x} dx \\ &= \frac{2}{3} (2\sqrt{2} - 1). \end{aligned}$$

$$2225. \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n}.$$

解 由于

$$\begin{aligned} &\lim_{n \rightarrow \infty} \ln \frac{\sqrt[n]{n!}}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\sum_{i=1}^n \ln i \right) - n \ln n \right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \ln \frac{i}{n} \cdot \frac{1}{n} = \int_0^1 \ln x dx^{**} \\ &= \lim_{\epsilon \rightarrow +0} \int_{\epsilon}^1 \ln x dx \\ &= \lim_{\epsilon \rightarrow +0} x(\ln x - 1) \Big|_{\epsilon}^1 = -1. \end{aligned}$$

从而

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = e^{-1} = \frac{1}{e}.$$

*) 参看后面 2388 题.

$$2226. \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{i=1}^n f \left(a + k \cdot \frac{b-a}{n} \right) \right].$$

$$\begin{aligned} \text{解} \quad &\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{i=1}^n f \left(a + k \frac{b-a}{n} \right) \right] \\ &= \int_0^1 f[a + (b-a)x] dx = \frac{1}{b-a} \int_a^b f(x) dx. \end{aligned}$$

弃掉高阶同等无穷小,求下列和的极限值:

$$2227. \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \sin \frac{\pi}{n^2} + \left(1 + \frac{2}{n} \right) \sin \frac{2\pi}{n^2} + \cdots + \left(1 + \frac{n-1}{n} \right) \sin \frac{(n-1)\pi}{n^2} \right].$$

解 由于对于一切 $k < n, 3 < n$ 有

$$\begin{aligned} 0 &\leq \frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2} \leq \operatorname{tg} \frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2} \\ &\leq \operatorname{tg} \frac{k\pi}{n^2} \left(1 - \cos \frac{k\pi}{n^2} \right) \\ &\leq \frac{\sin \frac{k\pi}{n^2}}{\cos \frac{k\pi}{n^2}} \left(1 - \cos \frac{k\pi}{n^2} \right) \\ &\leq \frac{2k\pi}{n^2} \left(1 - \cos \frac{\pi}{n} \right). \end{aligned}$$

从而,

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n-1} \left(1 + \frac{k}{n} \right) \left(\frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2} \right) \\ &\leq \sum_{k=1}^{n-1} \left(1 + \frac{k}{n} \right) \frac{2k\pi}{n^2} \left(1 - \cos \frac{\pi}{n} \right) \\ &\leq 2\pi \left(1 - \cos \frac{\pi}{n} \right) \rightarrow 0 (n \rightarrow +\infty), \end{aligned}$$

于是,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n} \right) \sin \frac{k\pi}{n^2} &= \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n} \right) \frac{k\pi}{n^2} \\ &\quad - \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n} \right) \left(\frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k\pi}{n} + \frac{k^2\pi}{n^2} \right) \end{aligned}$$

$$= \int_0^1 \pi(x + x_2) dx = \frac{5\pi}{6}.$$

$$2228. \lim_{n \rightarrow \infty} \sin \frac{\pi}{n} \cdot \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}}.$$

解 由于

$$\sin \frac{\pi}{n} = \frac{\pi}{n} (1 + \alpha_n),$$

式中 $\lim_{n \rightarrow \infty} \alpha_n = 0$.

于是,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sin \frac{\pi}{n} \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}} \\ &= \lim_{n \rightarrow \infty} (1 + \alpha_n) \frac{\pi}{n} \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}} \\ &= \left(\lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}} \right) \cdot \lim_{n \rightarrow \infty} (1 + \alpha_n) \\ &= \pi \int_0^1 \frac{dx}{2 + \cos \pi x} = \frac{2}{\sqrt{3}} \operatorname{arctg} \left[\frac{\operatorname{tg} \frac{\pi x}{2}}{\sqrt{3}} \right] \Big|_0^1 \\ &= \frac{\pi}{\sqrt{3}}. \end{aligned}$$

$$2229. \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sqrt{(nx + k)(nx + k + 1)}}{n^2} \quad (x > 0).$$

解 由于

$$0 \leq \sqrt{\left(x + \frac{k}{n}\right) \left(x + \frac{k+1}{n}\right)} - \left(x + \frac{k}{n}\right)$$

$$0 < \frac{1}{n} \sum_{k=1}^n 2^n - \sum_{k=1}^n \frac{2^{\frac{k}{n}}}{n + \frac{1}{k}} < \frac{1}{n^2} \sum_{k=1}^n 2^{\frac{k}{n}} < \frac{2}{n} \rightarrow 0$$

($n \rightarrow \infty$).

于是,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{n + \frac{1}{k}} \cdot 2^{\frac{k}{n}} \right] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n 2^{\frac{k}{n}}$$

$$= \int_0^1 2^x dx = \frac{1}{\ln 2}.$$

2231. 求: $\frac{d}{dx} \int_a^b \sin x^2 dx, \frac{d}{da} \int_a^b \sin x^2 dx,$
 $\frac{d}{db} \int_a^b \sin x^2 dx.$

解 $\frac{d}{dx} \int_a^b \sin x^2 dx = 0$

$$\frac{d}{da} \int_a^b \sin x^2 dx = - \frac{d}{da} \int_b^a \sin x^2 dx$$

$$= - \sin a^2$$

$$\frac{d}{db} \int_a^b \sin x^2 dx = \sin b^2.$$

2232. 求: (a) $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t^2} dt;$

(6) $\frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}};$

(B) $\frac{d}{dx} \int_{\sin x}^{\cos x} \cos(\pi t^2) dt.$

解 (a) $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t^2} dt$

$$= \left(\frac{d}{d(x^2)} \int_0^{x^2} \sqrt{1+t^2} dt \right) \cdot \frac{d}{dx}(x^2) \\ = 2x \cdot \sqrt{1+x^4};$$

$$(6) \quad \frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}} \\ = \frac{d}{dx} \int_{x^2}^0 \frac{dt}{\sqrt{1+t^4}} + \frac{d}{dx} \int_0^{x^3} \frac{dt}{\sqrt{1+t^4}} \\ = \frac{d}{dx}(x^3) \cdot \frac{d}{d(x^3)} \int_0^{x^3} \frac{dt}{\sqrt{1+t^4}} \\ - \frac{d}{dx}(x^2) \cdot \frac{d}{d(x^2)} \int_0^{x^2} \frac{dt}{\sqrt{1+t^4}} \\ = \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}};$$

$$(B) \quad \frac{d}{dx} \int_{\sin x}^{\cos x} \cos(\pi t^2) dt \\ = \frac{d}{dx} \int_{\sin x}^0 \cos(\pi t^2) dt \\ + \frac{d}{dx} \int_0^{\cos x} \cos(\pi t^2) dt \\ = - \frac{d(\sin x)}{dx} \cdot \frac{d}{d(\sin x)} \int_0^{\sin x} \cos(\pi t^2) dt \\ + \frac{d(\cos x)}{dx} \cdot \frac{d}{d(\cos x)} \int_0^{\cos x} \cos(\pi t^2) dt \\ = -\cos x \cdot \cos(\pi \sin^2 x) \\ - \sin x \cdot \cos(\pi \cos^2 x)^{*}) \\ = (\sin x - \cos x) \cdot \cos(\pi \sin^2 x).$$

$$*) \quad \cos(\pi \cos^2 x) = \cos(\pi - \pi \sin^2 x) \\ = -\cos(\pi \sin^2 x).$$

2233. 求:

$$(a) \lim_{x \rightarrow \infty} \frac{\int_0^x \cos x^2 dx}{x}; \quad (6) \lim_{x \rightarrow +\infty} \frac{\int_0^x (\operatorname{arc} \operatorname{tg} x)^2 dx}{\sqrt{x^2 + 1}};$$

$$(B) \lim_{x \rightarrow +\infty} \frac{\left(\int_0^x e^{x^2} dx \right)^2}{\int_0^x e^{2x^2} dx}.$$

解 (a) $\lim_{x \rightarrow \infty} \frac{\int_0^x \cos x^2 dx}{x} = \lim_{x \rightarrow 0} (\cos x^2) = 1;$

$$(6) \lim_{x \rightarrow +\infty} \frac{\int_0^x (\operatorname{arc} \operatorname{tg} x)^2 dx}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow +\infty} \frac{(\operatorname{arc} \operatorname{tg} x)^2}{\frac{x}{\sqrt{1 + x^2}}} \\ = \frac{\pi^2}{4};$$

$$(B) \lim_{x \rightarrow +\infty} \frac{\left(\int_0^x e^{x^2} dx \right)^2}{\int_0^x e^{2x^2} dx} \\ = \lim_{x \rightarrow +\infty} \frac{2e^{x^2} \cdot \int_0^x e^{x^2} dx}{e^{2x^2}} = \lim_{x \rightarrow +\infty} \frac{2 \int_0^x e^{x^2} dx}{e^{x^2}} \\ = \lim_{x \rightarrow +\infty} \frac{2e^{x^2}}{2xe^{x^2}} \\ = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

2234. 证明

$$\text{当 } x \rightarrow \infty \text{ 时, } \int_0^x e^{x^2} dx \sim \frac{1}{2x} e^{x^2}.$$

证 由于

$$\lim_{x \rightarrow \infty} \frac{\int_0^x e^{x^2} dx}{\frac{1}{2x} e^{x^2}} = \lim_{x \rightarrow \infty} \frac{e^{x^2}}{e^{x^2} \left(1 - \frac{1}{2x^2}\right)} = 1,$$

所以, 当 $x \rightarrow \infty$ 时,

$$\int_0^x e^{x^2} dx \sim \frac{1}{2x} e^{x^2}.$$

2235. 求:

$$\lim_{x \rightarrow +0} \frac{\int_0^{\sin x} \sqrt{\operatorname{tg} x} dx}{\int_0^{\operatorname{tg} x} \sqrt{\sin x} dx}.$$

解

$$\begin{aligned} & \lim_{x \rightarrow +0} \frac{\int_0^{\sin x} \sqrt{\operatorname{tg} x} dx}{\int_0^{\operatorname{tg} x} \sqrt{\sin x} dx} \\ &= \lim_{x \rightarrow +0} \frac{\sqrt{\operatorname{tg}(\sin x)} (\sin x)'}{\sqrt{\sin(\operatorname{tg} x)} (\operatorname{tg} x)'} \\ &= \lim_{x \rightarrow +0} \sqrt{\frac{\operatorname{tg}(\sin x)}{\sin x} \cdot \frac{\sin x}{\operatorname{tg} x} \cdot \frac{\operatorname{tg} x}{\sin(\operatorname{tg} x)}} \\ & \quad \cdot \lim_{x \rightarrow +0} \cos^3 x = 1. \end{aligned}$$

2236. 设 $f(x)$ 为连续正值函数, 证明当 $x \geq 0$ 时, 函数

$$\varphi(x) = \frac{\int_0^x t f(t) dt}{\int_0^x f(t) dt}$$

增加.

证 首先注意, $\lim_{x \rightarrow 0+} \varphi(x) = \lim_{x \rightarrow 0+} \frac{x f(x)}{f(x)} = 0$, 故若规定 $\varphi(0) = 0$, 则 $\varphi(x)$ 是 $x \geq 0$ 上的连续函数. 另外,

$$\begin{aligned}\varphi(x) &= \frac{1}{\left(\int_0^x f(t)dt\right)^2} \left\{ xf(x) \int_0^x f(t)dt \right. \\ &\quad \left. - f(x) \int_0^x tf(t)dt \right\} \\ &= \frac{f(x)}{\left(\int_0^x f(t)dt\right)^2} \cdot \int_0^x (x-t)f(t)dt \\ &> 0 \text{ (当 } x > 0 \text{ 时)},\end{aligned}$$

所以, $\varphi(x)$ 当 $x \geq 0$ 时是增加的.

2237. 求

$$\begin{aligned}\text{(a)} \quad & \int_0^2 f(x)dx, \text{ 设 } f(x) = \begin{cases} x^2, & \text{当 } 0 \leq x \leq 1, \\ 2-x, & \text{当 } 1 < x \leq 2; \end{cases} \\ \text{(6)} \quad & \int_0^1 f(x)dx, \text{ 设 } f(x) = \begin{cases} x, & \text{当 } 0 \leq x \leq t, \\ t \cdot \frac{1-x}{1-t}, & \text{当 } t \leq x \leq 1. \end{cases}\end{aligned}$$

$$\begin{aligned}\text{解} \quad \text{(a)} \quad & \int_0^2 f(x)dx = \int_0^1 x^2 dx + \int_1^2 (2-x)dx \\ &= \frac{5}{6}.\end{aligned}$$

$$\begin{aligned}\text{(6)} \quad & \int_0^1 f(x)dx = \int_0^t x dx + \int_t^1 t \cdot \frac{1-x}{1-t} dx \\ &= \frac{t}{2}.\end{aligned}$$

2238. 计算下列积分并把它们当作参数 α 的函数作出积分 $I=I(\alpha)$ 的图形. 设:

$$\text{(a)} \quad I = \int_0^1 x|x-\alpha|dx;$$

$$\text{(6)} \quad I = \int_0^\pi \frac{\sin^2 x}{1+2\alpha\cos x+\alpha^2} dx;$$

$$(B) \quad I = \int_0^x \frac{\sin x dx}{\sqrt{1 - 2a \cos x + a^2}}.$$

解 (a) 分三种情况:

1° 若 $a < 0$, 则

$$I = \int_0^1 x(x - a) dx = \frac{1}{3} - \frac{a}{2};$$

2° 若 $a > 1$, 则

$$I = \int_0^1 x(a - x) dx = \frac{a}{2} - \frac{1}{3};$$

3° 若 $0 \leq a \leq 1$, 则

$$\begin{aligned} I &= \int_0^a x(a - x) dx + \int_a^1 x(x - a) dx \\ &= \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3}. \end{aligned}$$

于是,

$$\int_0^1 x|x - a| dx = \begin{cases} \frac{1}{3} - \frac{a}{2}, & \text{当 } a < 0, \\ \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3}, & \text{当 } 0 \leq a \leq 1, \\ \frac{a}{2} - \frac{1}{3}, & \text{当 } a > 1 \end{cases} \text{图(4.9).}$$

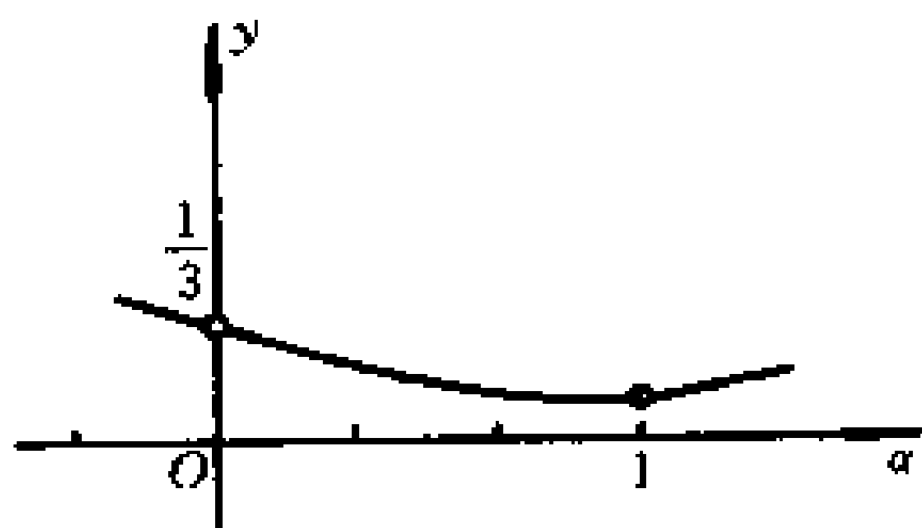


图 4.9

(6) 分两种情况:

1° 若 $|\alpha| \leq 1$, 则

$$\begin{aligned}
 I &= \int_0^\pi \frac{\sin^2 x}{1 + 2\cos x + \alpha^2} dx \\
 &= \frac{1}{4\alpha^2} \int_0^\pi \frac{4\alpha^2(1 - \cos^2 x) dx}{(1 + \alpha^2) + 2\alpha\cos x} \\
 &= \frac{1}{4\alpha^2} \int_0^\pi \frac{[(1 + \alpha^2)^2 - 4\alpha^2\cos^2 x] + [4\alpha^2 - (1 + \alpha^2)^2]}{(1 + \alpha^2) + 2\alpha\cos x} dx \\
 &= \frac{1}{4\alpha^2} \int_0^\pi [(1 + \alpha^2) - 2\alpha\cos x] dx - \frac{(1 - \alpha^2)^2}{4\alpha^2} \\
 &\quad \cdot \int_0^\pi \frac{dx}{(1 + \alpha^2) + 2\alpha\cos x} \\
 &= \frac{1}{4\alpha^2} [(1 + \alpha^2)x - 2\alpha\sin x] \Big|_0^\pi - \frac{(1 - \alpha^2)^2}{4\alpha^2} \\
 &\quad \cdot \frac{2}{1 - \alpha^2} \cdot \operatorname{arctg} \left[\sqrt{\frac{1 + \alpha^2 - 2\alpha}{1 + \alpha^2 + 2\alpha}} \operatorname{tg} \frac{x}{2} \right] \Big|_0^\pi \\
 &= \frac{(1 + \alpha^2)\pi}{4\alpha^2} - \frac{(1 - \alpha^2)\pi}{4\alpha^2} = \frac{\pi}{2}.
 \end{aligned}$$

2° 若 $|\alpha| > 1$, 则同上述情况类似有

$$\begin{aligned}
 I &= \frac{(1 + \alpha^2)\pi}{4\alpha^2} - \frac{(\alpha^2 - 1)^2}{4\alpha^2} \\
 &\quad \cdot \frac{2}{\alpha^2 - 1} \operatorname{arctg} \left[\sqrt{\frac{1 + \alpha^2 - 2\alpha}{1 + \alpha^2 + 2\alpha}} \operatorname{tg} \frac{x}{2} \right] \Big|_0^\pi \\
 &= \frac{(1 + \alpha^2)\pi}{4\alpha^2} - \frac{(\alpha^2 - 1)\pi}{4\alpha^2} \\
 &= \frac{\pi}{2\alpha^2}.
 \end{aligned}$$

于是,

$$\int_0^\pi \frac{\sin^2 x dx}{1 + 2\alpha\cos x + \alpha^2}$$

$$= \begin{cases} \frac{\pi}{2}, & \text{当 } |\alpha| \leq 1; \\ \frac{\pi}{2\alpha^2}, & \text{当 } |\alpha| > 1. \end{cases} \quad (\text{图 4.10})$$

*) 利用 2028 题(a)的结果.

$$\begin{aligned} (\text{B}) \quad & \int_0^{\pi} \frac{\sin x dx}{\sqrt{1 - 2a \cos x + a^2}} \\ &= \frac{1}{a} \sqrt{1 + a^2 - 2a \cos x} \Big|_0^{\pi} \\ &= \begin{cases} 2, & \text{当 } |\alpha| \leq 1, \\ \frac{2}{|\alpha|}, & \text{当 } |\alpha| > 1. \end{cases} \quad (\text{图 4.11}) \end{aligned}$$

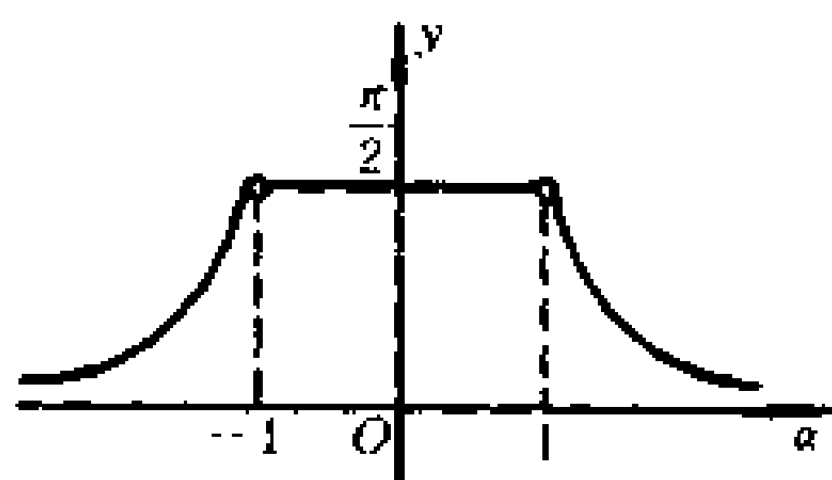


图 4.10

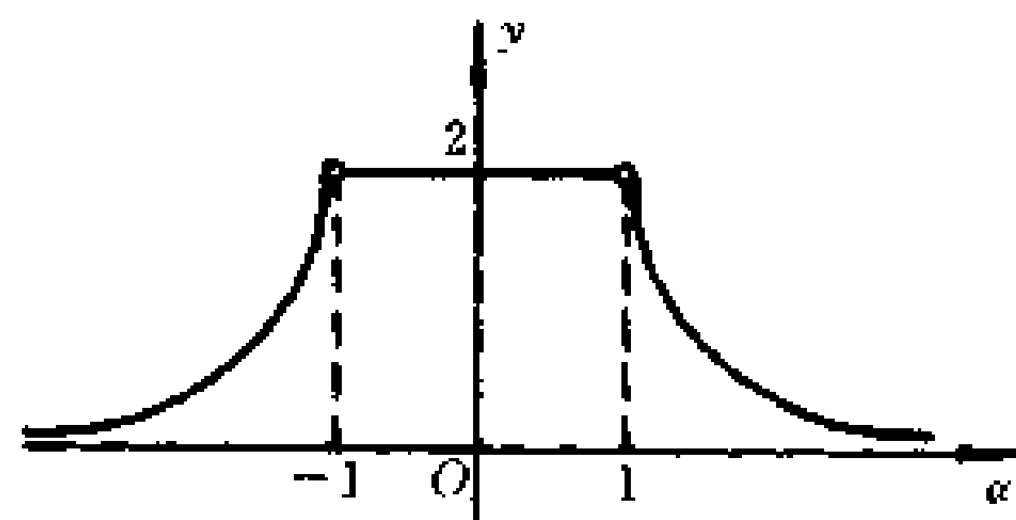


图 4.11

利用部分积分法的公式,求下列定积分:

2239. $\int_0^{\ln 2} x e^{-x} dx.$

$$\begin{aligned} \text{解} \quad \int_0^{\ln 2} x e^{-x} dx &= - \int_0^{\ln 2} x d(e^{-x}) \\ &= -x e^{-x} \Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x} dx \\ &= -\frac{1}{2} \ln 2 - e^{-x} \Big|_0^{\ln 2} \end{aligned}$$

$$= -\frac{1}{2}\ln 2 + \frac{1}{2} = \frac{1}{2}\ln \frac{e}{2}.$$

2240. $\int_0^{\pi} x \sin x dx.$

解 $\int_0^{\pi} x \sin x dx = -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx = \pi.$

2241. $\int_0^{2\pi} x^2 \cos x dx.$

解
$$\begin{aligned} \int_0^{2\pi} x^2 \cos x dx &= x^2 \sin x \Big|_0^{2\pi} - 2 \int_0^{2\pi} x \sin x dx \\ &= 2 \left(x \cos x \Big|_0^{2\pi} - \int_0^{2\pi} \cos x dx \right) \\ &= 4\pi. \end{aligned}$$

2242⁺. $\int_{\frac{1}{e}}^e |\lg x| dx.$

解
$$\begin{aligned} &\int_{\frac{1}{e}}^e |\lg x| dx \\ &= \int_{\frac{1}{e}}^1 (-\lg x) dx + \int_1^e \lg x dx \\ &= \left(-x \lg x \Big|_{\frac{1}{e}}^1 + \int_{\frac{1}{e}}^1 \frac{1}{\ln 10} dx \right) + x \lg x \Big|_1^e \\ &\quad - \int_1^e \frac{1}{\ln 10} dx \\ &= 2 \left(1 - \frac{1}{e} \right) \lg e. \end{aligned}$$

2243. $\int_0^1 \arccos x dx.$

解
$$\begin{aligned} &\int_0^1 \arccos x dx \\ &= x \arccos x \Big|_0^1 - \lim_{\epsilon \rightarrow +0} \int_0^{1-\epsilon} \frac{x}{\sqrt{1-x^2}} dx \end{aligned}$$

解 设 $t = \frac{1}{x+1}$, 则

$$\begin{aligned}
 & \int_0^{0.75} \frac{dx}{(x+1)\sqrt{x^2+1}} \\
 &= \int_{\frac{4}{7}}^1 \frac{dt}{\sqrt{2t^2-2t+1}} \\
 &= \frac{1}{\sqrt{2}} \ln(2t-1+\sqrt{2t^2-2t+2}) \Big|_{\frac{4}{7}}^1 \\
 &= \frac{1}{\sqrt{2}} \ln \frac{1+\sqrt{2}}{\frac{1}{7}+\sqrt{\frac{50}{49}}} = \frac{1}{\sqrt{2}} \ln \frac{7+7\sqrt{2}}{1+5\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}} \ln \frac{9+4\sqrt{2}}{7}.
 \end{aligned}$$

2248. $\int_0^{\ln 2} \sqrt{e^x-1} dx.$

解 设 $\sqrt{e^x-1}=t$, 则

$$\begin{aligned}
 & \int_0^{\ln 2} \sqrt{e^x-1} dx \\
 &= 2 \int_0^1 \frac{t^2 dt}{1+t^2} = 2(t - \arctg t) \Big|_0^1 = 2 - \frac{\pi}{2}.
 \end{aligned}$$

2249. $\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx.$

解 设 $\sqrt{x}=t$, 则

$$\begin{aligned}
 & \int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx \\
 &= 2 \int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} dt = (\arcsin t)^2 \Big|_0^1 = \frac{\pi^2}{4}.
 \end{aligned}$$

2250. 假设 $x - \frac{1}{x} = t$, 来计算积分

$$\int_{-1}^1 \frac{1+x^2}{1+x^4} dx.$$

解 由于被积函数是偶函数, 于是,

$$\begin{aligned} & \int_{-1}^1 \frac{1+x^2}{1+x^4} dx \\ &= 2 \int_0^1 \frac{1+x^2}{1+x^4} dx = \lim_{N \rightarrow +\infty} 2 \int_N^0 \frac{dt}{t^2+2} \\ &= \lim_{N \rightarrow +\infty} \sqrt{2} \operatorname{arctg} \frac{t}{\sqrt{2}} \Big|_N^0 = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

2251. 设:

$$(a) \int_{-1}^1 dx, \quad t = x^{\frac{2}{3}};$$

$$(b) \int_{-1}^1 \frac{dx}{1+x^2}, \quad x = \frac{1}{t};$$

$$(c) \int_0^\pi \frac{dx}{1+\sin^2 x}, \quad \operatorname{tg} x = t.$$

说明为甚么用 $\varphi(t)$ 代换 x 会引致不正确的结果.

解 (a) $\int_{-1}^1 dx = 2$. 但若作代换 $t = x^{\frac{2}{3}}$, 则得

$$\int_{-1}^1 dx = \pm \frac{3}{2} \int_1^1 t^{\frac{1}{2}} dt = 0.$$

其错误在于代换 $t = x^{\frac{2}{3}}$ 的反函数 $x = \pm t^{\frac{3}{2}}$ 不是单值的.

(b) $\int_{-1}^1 \frac{dx}{1+x^2} = \operatorname{arctg} x \Big|_{-1}^1 = \frac{\pi}{2}$. 但若作代换 $x = \frac{1}{t}$, 则得

$$\int_{-1}^1 \frac{dx}{1+x^2} = - \int_{-1}^1 \frac{dt}{1+t^2},$$

于是得出错误的结果: $\int_{-1}^1 \frac{dx}{1+x^2} = 0$.

其错误在于 $x = \frac{1}{t}$, 当 $t = 0$ (0 属于 $[-1, 1]$) 时不连续.

(B) $\int_0^{\pi} \frac{dx}{1 + \sin^2 x}$ 大于零, 但若作代换 $t = \operatorname{tg} x$, 则得

$$\int_0^{\pi} \frac{dx}{1 + \sin^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} x) \Big|_0^{\pi} = 0.$$

其错误在于 $t = \operatorname{tg} x$ 在 $x = \frac{\pi}{2}$ 处不连续.

2252. 在积分

$$\int_0^3 x \sqrt[3]{1 - x^2} dx$$

中, 令 $x = \sin t$ 是否可以?

解 不可以, 因为 $\sin t = x$ 不可能大于 1.

2253. 于积分 $\int_0^1 \sqrt{1 - x^2} dx$ 中, 当作变数的代换 $x = \sin t$ 时, 可否取数 π 和 $\frac{\pi}{2}$ 作为新的上下限?

解 可以. 因为满足定积分换元的条件. 事实上,

$$\begin{aligned} \int_0^1 \sqrt{1 - x^2} dx &= \int_{\pi}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t} d(\sin t) \\ &= \int_{\pi}^{\frac{\pi}{2}} |\cos t| \cos t dt = - \int_{\pi}^{\frac{\pi}{2}} \cos^2 t dt \\ &= \left(\frac{t}{2} + \frac{\sin 2t}{4} \right) \Big|_{\frac{\pi}{2}}^{\pi} = \frac{\pi}{4}. \end{aligned}$$

2254. 证明: 若函数 $f(x)$ 于闭区间 $[a, b]$ 内连续, 则

$$\int_a^b f(x) dx = (b - a) \int_0^1 f[a + (b - a)x] dx.$$

证 设 $x=a+(b-a)t$, 则 $dx=(b-a)dt$.

代入得

$$\int_a^b f(x)dx = (b-a) \int_0^1 f[a+(b-a)t]dt,$$

即

$$\int_a^b f(x)dx = (b-a) \int_0^1 f[a+(b-a)x]dx.$$

2255. 证明: 等式

$$\int_0^a x^3 f(x^2)dx = \frac{1}{2} \int_0^{a^2} xf(x)dx \quad (a>0).$$

证 设 $x=\sqrt{t}$, 则

$$\begin{aligned} & \int_0^a x^3 f(x^2)dx \\ &= \int_0^{a^2} t^{\frac{3}{2}} f(t) \cdot \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^{a^2} tf(t)dt. \end{aligned}$$

即

$$\int_0^a x^3 f(x^2)dx = \frac{1}{2} \int_0^{a^2} xf(x)dx.$$

2256. 设 $f(x)$ 为闭区间 $[A, B] \supset [a, b]$ 上的连续函数, 当

$A-a < x < B-b$ 时, 求 $\frac{d}{dx} \int_a^b f(x+y)dy$.

$$\begin{aligned} \text{解} \quad & \frac{d}{dx} \int_a^b f(x+y)dy \\ &= \frac{d}{dx} \int_{a+x}^{b+x} f(y)dy = f(b+x) - f(a+x). \end{aligned}$$

2257. 证明: 若函数 $f(x)$ 于闭区间 $[0, 1]$ 上连续, 则

$$(a) \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx;$$

$$(b) \int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)dx.$$

证(a) 设 $\frac{\pi}{2} - t = x$, 则 $dx = -dt$, 且

$$f(\sin x) = f(\cos t).$$

代入得

$$\begin{aligned} \int_0^{\frac{\pi}{2}} f(\sin x) dx &= - \int_{\frac{\pi}{2}}^0 f(\cos t) dt \\ &= \int_0^{\frac{\pi}{2}} f(\cos t) dt, \end{aligned}$$

即

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

(b) 设 $\pi - t = x$, 则 $dx = -dt$, 且

$$xf(\sin x) = (\pi - t)f(\sin t).$$

代入得

$$\begin{aligned} \int_0^{\pi} xf(\sin x) dx &= - \int_{\pi}^0 (\pi - t)f(\sin t) dt \\ &= \pi \int_0^{\pi} f(\sin t) dt - \int_0^{\pi} tf(\sin t) dt, \end{aligned}$$

即

$$\int_0^{\pi} xf(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx.$$

2258. 证明: 若函数 $f(x)$ 于闭区间 $[-l, l]$ 上连续, 则

(1) 若函数 $f(x)$ 为偶函数时,

$$\int_{-l}^l f(x) dx = 2 \int_0^l f(x) dx;$$

(2) 若函数 $f(x)$ 为奇函数时,

$$\int_{-l}^l f(x) dx = 0.$$

给出这些事实的几何解释.

证 (1) 由于 $f(x)$ 为偶函数, 即 $f(x) = f(-x)$,
 $(x \in [-l, l])$. 于是设 $x = -t$, 则有

$$\begin{aligned}\int_{-l}^l f(x) dx &= \int_{-l}^0 f(x) dx + \int_0^l f(x) dx \\ &= - \int_{-l}^0 f(-x) d(-x) + \int_0^l f(x) dx \\ &= - \int_l^0 f(t) dt + \int_0^l f(x) dx \\ &= \int_0^l f(t) dt + \int_0^l f(x) dx = 2 \int_0^l f(x) dx.\end{aligned}$$

其几何解释如下:

由于 $f(x) = f(-x)$, 故图形关于 Oy 轴对称. 于是由曲线 $y = f(x)$, 直线 $x = -l$ 及 $x = l$ 所围成图形的面积为由曲线 $y = f(x)$, 直线 $x = 0$ 及 $x = l$ 所围成的图形的面积 S 的两倍(图 4.12).

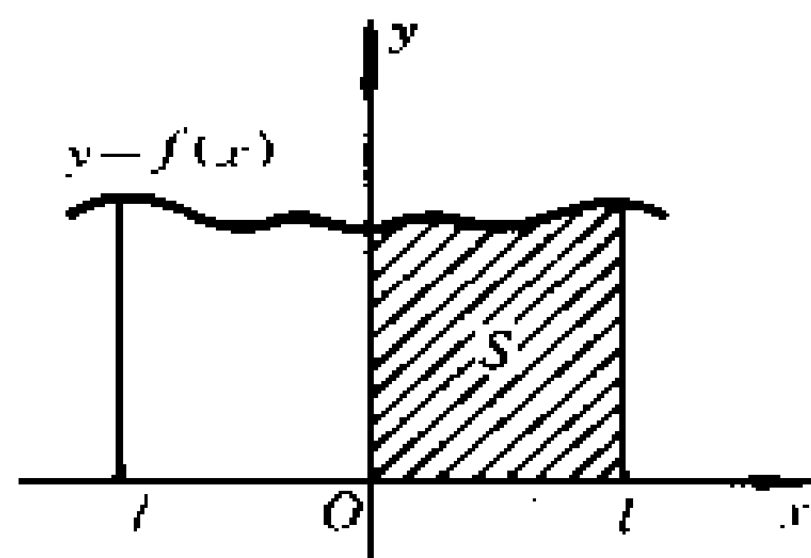


图 4.12

(2) 由于 $f(x) = -f(-x)$, 设 $x = -t$, 则

$$\begin{aligned}\int_{-l}^l f(x) dx &= - \int_{-l}^0 f(-x) dx + \int_0^l f(x) dx \\ &= \int_l^0 f(t) dt + \int_0^l f(x) dx = 0.\end{aligned}$$

其几何解释如下:

由于 $f(x) = -f(-x)$, 故图形关于原点对称. 于是

由 $-l$ 到 0 之间所围之面积, 与由 0 到 l 之间所围成之面积绝对值相等, 符号相反, 故其面积的代数和为零 (图 4.13).

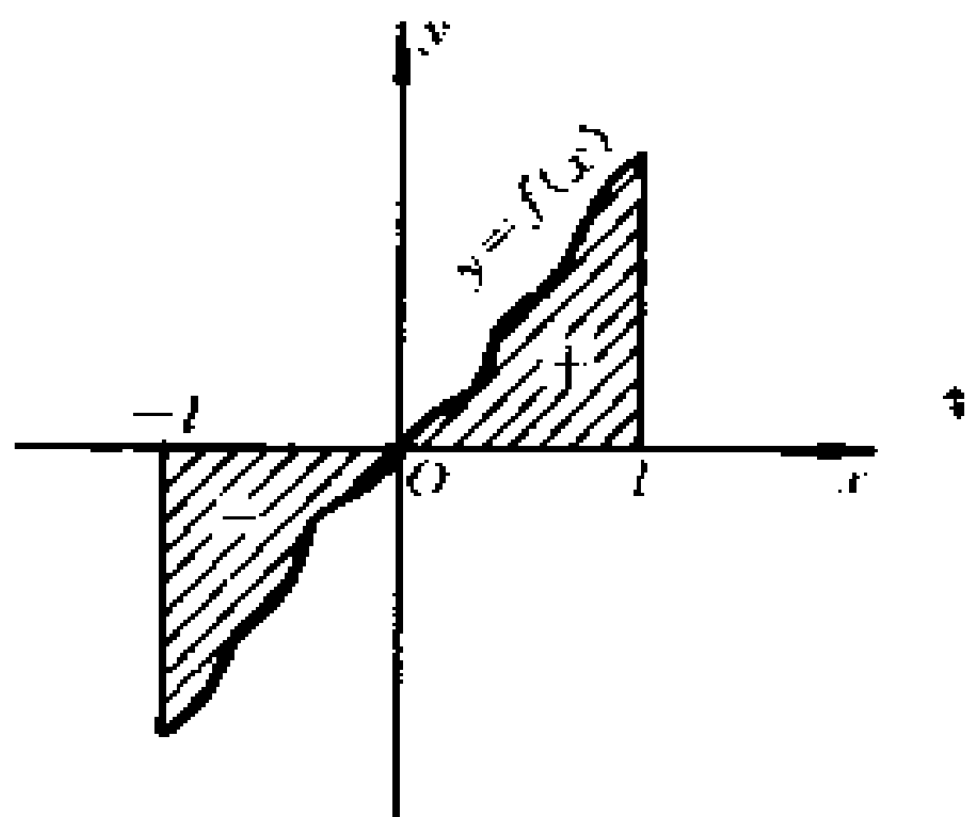


图 4.13

2259. 证明: 偶函数的原函数中之一为奇函数, 而奇函数的一切原函数皆为偶函数.

证 设 $f(x)$ 在 $[-l, l]$ 上定义^{*}, 且 $F(x)$ 是 $f(x)$ 的一个原函数, 当 $f(-x) = f(x)$ 时, 由于

$$f(x) = \frac{d}{dx}F(x) \text{ 及 } f(-x) = -\frac{d}{dx}F(-x),$$

故有 $\frac{d}{dx}[F(x) + F(-x)] = 0$. 从而可得

$$F(x) + F(-x) = C_1, \text{ 且 } C_1 = 2F(0).$$

于是, $f(x)$ 有一个原函数 $F(x) - F(0)$ 是奇函数.

当 $f(-x) = -f(x)$ 时, 类似地可得

$$F(x) - F(-x) = C_2, \text{ 且 } C_2 = 0.$$

于是, $F(-x) = F(x)$, 即 $f(x)$ 的任一原函数 $F(x) + C$ (C 为任意常数) 也为偶函数.

*) 如果 $f(x)$ 在 $[-l, l]$ 上可积, 则由

$$F_c(x) = \int_0^x f(t)dt + C \quad (C \text{ 是任意常数})$$

也可获证, 其中 $F_c(x)$ 为 $f(x)$ 的全部原函数.

2260. 引入新变数

$$t = x + \frac{1}{x}.$$

来计算积分 $\int_{\frac{1}{2}}^2 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx$.

解 设 $t = x + \frac{1}{x}$, 则

$$t^2 - 4 = \left(x - \frac{1}{x}\right)^2, x = \frac{1}{2}(t \pm \sqrt{t^2 - 4}).$$

于是,

$$\begin{aligned} & \int_{\frac{1}{2}}^2 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx \\ &= \int_1^2 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx \\ & \quad + \int_{\frac{1}{2}}^1 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx \\ &= \int_2^{\frac{5}{2}} (1 + \sqrt{t^2 - 4}) e^t d\left[\frac{1}{2}(t + \sqrt{t^2 - 4})\right] \\ & \quad + \int_{\frac{5}{2}}^2 (1 - \sqrt{t^2 - 4}) e^t d\left[\frac{1}{2}(t - \sqrt{t^2 - 4})\right] \\ &= \frac{1}{2} \int_2^{\frac{5}{2}} (1 + \sqrt{t^2 - 4}) e^t \left(1 + \frac{t}{\sqrt{t^2 - 4}}\right) dt \\ & \quad - \frac{1}{2} \int_{\frac{5}{2}}^2 (1 - \sqrt{t^2 - 4}) e^t \left(1 - \frac{t}{\sqrt{t^2 - 4}}\right) dt \\ &= \int_2^{\frac{5}{2}} e^t \left(\sqrt{t^2 - 4} + \frac{t}{\sqrt{t^2 - 4}}\right) dt \\ &= \int_2^{\frac{5}{2}} [\sqrt{t^2 - 4} d(e^t) + e^t d\sqrt{t^2 - 4}] \\ &= (\sqrt{t^2 - 4}) e^t \Big|_2^{\frac{5}{2}} = \frac{3}{2} e^{\frac{5}{2}}. \end{aligned}$$

2261. 于积分

$$\int_0^{2\pi} f(x) \cos x dx.$$

中实行变数代换 $\sin x = t$.

解
$$\begin{aligned} \int_0^{2\pi} f(x) \cos x dx &= \int_0^{\frac{\pi}{2}} f(x) \cos x dx + \int_{\frac{\pi}{2}}^{\pi} f(x) \cos x dx \\ &\quad + \int_{\pi}^{\frac{3\pi}{2}} f(x) \cos x dx + \int_{\frac{3\pi}{2}}^{2\pi} f(x) \cos x dx. \end{aligned}$$

在右端的第一个积分中, 设 $\sin x = t$; 第二、第三个积分中, 设 $\sin(\pi - x) = t$; 第四个积分中, 设 $\sin(2\pi - x) = -t$, 代入得

$$\begin{aligned} &\int_0^{2\pi} f(x) \cos x dx \\ &= \int_0^1 [f(\arcsin t) - f(\pi - \arcsin t)] dt \\ &\quad + \int_{-1}^0 [f(2\pi + \arcsin t) - f(\pi - \arcsin t)] dt. \end{aligned}$$

2262. 计算积分

$$\int_{e^{-2\pi n}}^1 \left| \left[\cos \left(\ln \frac{1}{x} \right) \right]' \right| dx,$$

式中 n 为自然数.

证 $\left[\cos \left(\ln \frac{1}{x} \right) \right]' = \frac{\sin(-\ln x)}{x}$. 设 $x = e^{-t}$,

则 $dx = -e^{-t} dt$, $\frac{\sin(-\ln x)}{x} = \frac{\sin t}{e^{-t}} = e^t \sin t$.

代入得

$$\int_{e^{-2\pi n}}^1 \left| \left[\cos \left(\ln \frac{1}{x} \right) \right]' \right| dx = \int_0^{2\pi n} |\sin t| dt$$

2265. 证明:若 $f(x)$ 为定义在 $-\infty < x < +\infty$ 而周期为 T 的连续的周期函数,则

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx.$$

式中 a 为任意的数.

$$\begin{aligned} \text{证} \quad & \int_a^{a+T} f(x)dx \\ &= \int_a^0 f(x)dx + \int_a^T f(x)dx + \int_T^{a+T} f(x)dx. \end{aligned}$$

对上述等式右端的第三个积分,设 $x - T = t$,则

$$\int_a^{a+T} f(x)dx = \int_0^a f(t+T)dt = \int_0^a f(t)dt.$$

于是,

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx.$$

2266. 证明: 当 n 为奇数时,函数

$$F(x) = \int_0^x \sin^n x dx \text{ 及 } G(x) = \int_0^x \cos^n x dx$$

为以 2π 为周期的周期函数;而当 n 为偶数时,则其中的任何一个皆为线性函数与周期函数的和.

证 当 n 为奇数时, $\sin^n x$ 是奇函数,而且是以 2π 为周期的函数. 于是,

$$\begin{aligned} F(x+2\pi) &= \int_0^{x+2\pi} \sin^n x dx \\ &= \int_0^{2\pi} \sin^n x dx + \int_{2\pi}^{2\pi+x} \sin^n x dx \\ &= \int_{-\pi}^{\pi} \sin^n(\pi-x) dx + \int_0^x \sin^n x dx \\ &= 0 + \int_0^x \sin^n x dx = F(x) \end{aligned}$$

及

$$\begin{aligned}G(x+2\pi) &= G(x) + \int_0^{2\pi} \cos^n x dx \\&= G(x) + \int_0^\pi \cos^n x dx + \int_\pi^{2\pi} \cos^n x dx \\&= G(x) + \int_0^\pi \cos^n x dx + \int_0^\pi \cos^n(x+\pi) dx \\&= G(x),\end{aligned}$$

从而得知: $F(x)$ 和 $G(x)$ 都是以 2π 为周期的周期函数. 当 n 为偶数时, 显然有

$$\begin{aligned}F(x+2\pi) &= F(x) + \int_0^{2\pi} \sin^n x dx, \\G(x+2\pi) &= G(x) + \int_0^{2\pi} \cos^n x dx.\end{aligned}$$

但因

$$\int_0^{2\pi} \sin^n x dx = \int_0^{2\pi} \cos^n x dx = a > 0,$$

所以, $F(x)$ 、 $G(x)$ 都不是 2π 为周期的周期函数. 设

$$F_1(x) = F(x) - \frac{a}{2\pi}x,$$

则

$$\begin{aligned}F_1(x+2\pi) &= F(x+2\pi) - \frac{a}{2\pi}(x+2\pi) \\&= F(x) + a - \frac{a}{2\pi}x - a \\&= F(x) - \frac{a}{2\pi}x = F_1(x).\end{aligned}$$

即 $F_1(x)$ 是以 2π 为周期的函数, 而

$$F(x) = F_1(x) + \frac{a}{2\pi}x.$$

所以, $F(x)$ 为周期函数与线性函数之和.

同理, 可以证明 $G(x)$ 也是周期函数与线性函数之和.

2267. 证明: 函数

$$F(x) = \int_{x_0}^x f(x) dx$$

(式中 $f(x)$ 为具周期 T 的连续的周期函数) 在一般的情形下是线性函数与周期函数之和.

证 因为 $F(x) = \int_{x_0}^x f(x) dx$, 所以

$$F(x+T) - F(x) = \int_x^{x+T} f(x) dx.$$

又因 $f(x)$ 是一周期为 T 的连续函数, 所以

$$\int_x^{x+T} f(x) dx = \int_{x_0}^{x_0+T} f(x) dx = K.$$

于是, $F(x+T) - F(x) = K$.

如果 $K = 0$, 则 $F(x)$ 为一周期函数.

如果 $K \neq 0$, 可考虑函数 $\varphi(x) = F(x) - \frac{K}{T}x$,

则因

$$\begin{aligned}\varphi(x+T) &= F(x+T) - \frac{K}{T}(x+T) \\ &= F(x+T) - \frac{K}{T}x - K \\ &= F(x) - \frac{K}{T}x = \varphi(x),\end{aligned}$$

所以, $\varphi(x)$ 也为一周期函数, 从而

$$F(x) = \varphi(x) + \frac{K}{T}x,$$

即 $F(x)$ 是线性函数与周期等于 T 的周期函数之和.

计算下列积分:

$$2268. \int_0^1 x(2-x^2)^{12} dx.$$

$$\begin{aligned} \text{解} \quad & \int_0^1 x(2-x^2)^{12} dx \\ &= -\frac{1}{26}(2-x^2)^{13} \Big|_0^1 = 315 \frac{1}{26}. \end{aligned}$$

$$2269. \int_{-1}^1 \frac{x dx}{x^2+x+1}.$$

$$\begin{aligned} \text{解} \quad & \int_{-1}^1 \frac{x dx}{x^2+x+1} \\ &= \frac{1}{2} \int_{-1}^1 \frac{2x+1}{x^2+x+1} dx - \frac{1}{2} \int_{-1}^1 \frac{dx}{\frac{3}{4} + \left(x + \frac{1}{2}\right)^2} \\ &= \frac{1}{2} \ln(x^2+x+1) \Big|_{-1}^1 - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \Big|_{-1}^1 \\ &= \frac{1}{2} \ln 3 - \frac{\pi}{2\sqrt{3}}. \end{aligned}$$

$$2270^+. \int_1^e (x \ln x)^2 dx.$$

$$\begin{aligned} \text{解} \quad & \int_1^e (x \ln x)^2 dx \\ &= x^3 \ln^2 x \Big|_1^e - 2 \int_1^e x^2 \ln x \cdot (1 + \ln x) dx \\ &= e^3 - 2 \int_1^e x^2 \ln x dx - 2 \int_1^e (x \ln x)^2 dx. \end{aligned}$$

移项合并得

$$\begin{aligned} \int_1^e (x \ln x)^2 dx &= \frac{e^3}{3} - \frac{2}{3} \int_1^e x^2 \ln x dx \\ &= \frac{e^3}{3} - \left(\frac{2}{9} x^3 \ln x - \frac{2}{27} x^3 \right) \Big|_1^e \end{aligned}$$

$$= \frac{5}{27}e^3 - \frac{2}{27}.$$

$$2271. \int_1^9 x \sqrt[3]{1-x} dx.$$

解 设 $\sqrt[3]{1-x} = t$, 则

$$\int_1^9 x \sqrt[3]{1-x} dx$$

$$= -3 \int_0^{-2} (t^3 - t^6) dt = -66 \frac{6}{7}.$$

$$2272^+. \int_{-2}^{-1} \frac{dx}{x \sqrt{x^2-1}}.$$

解 设 $x = \frac{1}{t}$, 则

$$\int_{-2}^{-1} \frac{dx}{x \sqrt{x^2-1}}$$

$$= \int_{-\frac{1}{2}}^{-1} \frac{dt}{\sqrt{1-t^2}} = \arcsin t \Big|_{-\frac{1}{2}}^{-1} = -\frac{\pi}{3}.$$

$$2273. \int_0^1 x^{15} \sqrt{1+3x^8} dx.$$

解 设 $1+3x^8 = t$, 则 $24x^7 dx = dt$, $x^8 = \frac{1}{3}(t-1)$.

于是,

$$\int_0^1 x^{15} \sqrt{1+3x^8} dx$$

$$= \frac{1}{72} \int_1^4 (t-1) t^{\frac{1}{2}} dt$$

$$= \frac{29}{270}.$$

$$2274. \int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx.$$

$$\begin{aligned}
& \text{解} \quad \int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx \\
&= x \arcsin \sqrt{\frac{x}{1+x}} \Big|_0^3 - \int_0^3 \frac{\sqrt{x} dx}{2(1+x)} \\
&= \pi - \int_0^{\sqrt{3}} \frac{t^2 dt}{1+t^2} \quad (*) = \pi - (t - \operatorname{arctg} t) \Big|_0^{\sqrt{3}} \\
&= \frac{4\pi}{3} - \sqrt{3}.
\end{aligned}$$

*) 设 $\sqrt{x} = t$.

$$2275. \quad \int_0^{2\pi} \frac{dx}{(2 + \cos x)(3 + \cos x)}.$$

$$\begin{aligned}
& \text{解} \quad \int_0^{2\pi} \frac{dx}{(2 + \cos x)(3 + \cos x)} \\
&= \int_0^{2\pi} \frac{dx}{2 + \cos x} - \int_0^{2\pi} \frac{dx}{3 + \cos x} \\
&= \int_0^{\pi} \frac{dx}{2 + \cos x} + \int_0^{\pi} \frac{dx}{2 - \cos x} \\
&\quad - \int_0^{2\pi} \frac{dx}{3 + \cos x} \\
&= 4 \int_0^{\pi} \frac{dx}{4 - \cos^2 x} - 6 \int_0^{\pi} \frac{dx}{9 - \cos^2 x} \\
&= 8 \int_0^{\frac{\pi}{2}} \frac{dx}{4 \sin^2 x + 3 \cos^2 x} \\
&\quad - 12 \int_0^{\frac{\pi}{2}} \frac{dx}{9 \sin^2 x + 8 \cos^2 x} \\
&= 8 \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{2 \operatorname{tg} x}{\sqrt{3}} \Big|_0^{\frac{\pi}{2}} \\
&\quad - 12 \frac{1}{3 \cdot \sqrt{8}} \operatorname{arctg} \frac{3 \operatorname{tg} x}{\sqrt{8}} \Big|_0^{\frac{\pi}{2}}
\end{aligned}$$

$$= \pi \left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{2}} \right).$$

2276. $\int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x}.$

解 $\int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x}$
 $= 8 \int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$
 $= \frac{1}{\sqrt{2}} \operatorname{arctg} \left(\frac{\operatorname{tg} 2x}{\sqrt{2}} \right) \Big|_0^{2\pi} = 2\pi \sqrt{2}.$

*) 利用 2035 题的结果.

2277. $\int_0^{\frac{\pi}{2}} \sin x \sin 2x \sin 3x dx.$

解 $\sin x \sin 2x \sin 3x$
 $= \frac{1}{2} (\cos 2x - \cos 4x) \cdot \sin 2x$
 $= \frac{1}{4} \sin 4x - \frac{1}{4} (\sin 6x - \sin 2x).$

于是,

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sin x \sin 2x \sin 3x dx \\ &= \left(-\frac{1}{16} \cos 4x + \frac{1}{24} \cos 6x - \frac{1}{8} \cos 2x \right) \Big|_0^{\frac{\pi}{2}} \\ &= \frac{1}{6}. \end{aligned}$$

2278. $\int_0^{\pi} (x \sin x)^2 dx.$

解 $\int_0^{\pi} (x \sin x)^2 dx$
 $= \frac{1}{2} \int_0^{\pi} x^2 (1 - \cos 2x) dx$

$$\begin{aligned}
&= \frac{1}{6}x^3 \Big|_0^\pi - \frac{1}{2} \int_0^\pi x^2 \cos 2x dx \\
&= \frac{\pi^3}{6} - \frac{x^2}{4} \sin 2x \Big|_0^\pi + \frac{1}{2} \int_0^\pi x \sin 2x dx \\
&= \frac{\pi^3}{6} - \frac{x}{4} \cos 2x \Big|_0^\pi + \frac{1}{4} \int_0^\pi \cos 2x dx \\
&= \frac{\pi^3}{6} - \frac{\pi}{4}.
\end{aligned}$$

2279. $\int_0^\pi e^x \cos^2 x dx.$

解 $\int_0^\pi e^x \cos^2 x dx$

$$\begin{aligned}
&= \int_0^\pi \frac{e^x (1 + \cos 2x)}{2} dx \\
&= \frac{e^x}{2} + \frac{e^x}{10} (\cos 2x + 2 \sin 2x)^{*}) \Big|_0^\pi \\
&= \frac{3}{5} (e^\pi - 1).
\end{aligned}$$

*) 利用 1828 题的结果.

2280. $\int_0^{\ln 2} \text{sh}^4 x dx.$

解 $\int_0^{\ln 2} \text{sh}^4 x dx$

$$\begin{aligned}
&= \int_0^{\ln 2} \text{sh}^2 x \cdot (\text{ch}^2 x - 1) dx \\
&= \frac{1}{4} \int_0^{\ln 2} \text{sh}^2 2x dx - \int_0^{\ln 2} \text{sh}^2 x dx \\
&= \frac{1}{4} \left(-\frac{x}{2} + \frac{1}{8} \text{sh} 4x \right)^{*}) \Big|_0^{\ln 2} \\
&\quad - \left(-\frac{x}{2} + \frac{1}{4} \text{sh} 2x \right)^{*}) \Big|_0^{\ln 2} \\
&= \frac{3}{8} \ln 2 - \frac{225}{1024}.
\end{aligned}$$

*) 利用 1761 题的结果.

利用递推公式来计算下列依赖于取正整数值的参数 n 的积分.

$$2281. I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx.$$

$$\begin{aligned} \text{解} \quad I_n &= - \int_0^{\frac{\pi}{2}} \sin^{n-1} x d(\cos x) \\ &= - \sin^{n-1} x \cdot \cos x \Big|_0^{\frac{\pi}{2}} \\ &\quad + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx \\ &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x dx, \end{aligned}$$

移项合并得

$$I_n = \frac{n-1}{n} I_{n-2}.$$

利用上述递推公式即可求得

$$I_n = \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & \text{若 } n = 2k; \\ \frac{(2k)!!}{(2k+1)!!}, & \text{若 } n = 2k+1. \end{cases}$$

$$2282. I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx.$$

解 设 $\frac{\pi}{2} - x = t$, 则 $dx = -dt$, 且

$$\cos x = \cos\left(\frac{\pi}{2} - t\right) = \sin t.$$

代入得

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t dt.$$

解 设 $x = \sin t$, 代入得

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t dt.$$

因此, 与 2281 题的结果相同.

$$2286. I_n = \int_0^1 x^m (\ln x)^n dx.$$

$$\begin{aligned} \text{解 } I_n &= \frac{1}{m+1} x^{m+1} \ln^n x \Big|_0^1 \\ &\quad - \frac{n}{m+1} \int_0^1 x^m (\ln x)^{n-1} dx. \end{aligned}$$

于是,

$$\begin{aligned} I_n &= -\frac{n}{m+1} I_{n-1} \\ &= \left(-\frac{n}{m+1} \right) \left(-\frac{n-1}{m+1} \right) \cdots \left(-\frac{1}{m+1} \right) I_0 \\ &= (-1)^n \cdot \frac{n!}{(m+1)^{n+1}}. \end{aligned}$$

$$2287. I_n = \int_0^{\frac{\pi}{4}} \left(\frac{\sin x - \cos x}{\sin x + \cos x} \right)^{2n+1} dx.$$

$$\begin{aligned} \text{解 } I_n &= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n+1} \left(x - \frac{\pi}{4} \right) dx \\ &= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-1} \left(x - \frac{\pi}{4} \right) \cdot \left[\sec^2 \left(x - \frac{\pi}{4} \right) - 1 \right] dx \\ &= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-1} \left(x - \frac{\pi}{4} \right) d \left[\operatorname{tg} \left(x - \frac{\pi}{4} \right) \right] - I_{n-1} \\ &= -\frac{1}{2n} - I_{n-1}, \end{aligned}$$

即

$$I_n = -\frac{1}{2n} - I_{n-1}$$

递推之,得

$$I_n = -\frac{1}{2n} + \frac{1}{2(n-1)} - \frac{1}{2(n-2)} + \dots \\ + (-1)^n \cdot \frac{1}{2} + (-1)^n I_0.$$

但

$$I_0 = \int_0^{\frac{\pi}{4}} \operatorname{tg}\left(x - \frac{\pi}{4}\right) dx \\ = -\ln \left| \cos\left(x - \frac{\pi}{4}\right) \right| \Big|_0^{\frac{\pi}{4}} \\ = \ln \frac{\sqrt{2}}{2} = -\ln \sqrt{2},$$

于是,

$$I_n = (-1)^n \left\{ -\ln \sqrt{2} + \frac{1}{2} \left[1 - \frac{1}{2} + \dots \right. \right. \\ \left. \left. + (-1)^{n-1} \frac{1}{n} \right] \right\}.$$

设 $f(x) = f_1(x) + if_2(x)$ 是实变数 x 的复函数, 其中 $f_1(x) = \operatorname{Re} f(x)$, $f_2(x) = \operatorname{Im} f(x)$ 及 $i = \sqrt{-1}$, 则按定义有:

$$\int f(x) dx = \int f_1(x) dx + i \int f_2(x) dx.$$

显而易见

$$\operatorname{Re} \int f(x) dx = \int \operatorname{Re} f(x) dx.$$

$$\text{及} \quad \operatorname{Im} \int f(x) dx = \int \operatorname{Im} f(x) dx.$$

2288. 利用尤拉氏公式

$$e^{ix} = \cos x + i \sin x,$$

证明:

$$\int_0^{2\pi} e^{inx} e^{-imx} dx = \begin{cases} 0, & \text{若 } m \neq n, \\ 2\pi, & \text{若 } m = n. \end{cases}$$

(n 及 m 为整数).

证 当 $m = n$ 时,

$$\int_0^{2\pi} e^{inx} e^{-imx} dx = \int_0^{2\pi} dx = 2\pi.$$

当 $m \neq n$ 时,

$$\begin{aligned} & \int_0^{2\pi} e^{inx} e^{-imx} dx \\ &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx) dx \\ &= \int_0^{2\pi} \cos(m-n)x dx - i \int_0^{2\pi} \sin(m-n)x dx = 0. \end{aligned}$$

2289. 证明

$$\int_a^b e^{(\alpha+i\beta)x} dx = \frac{e^{b(\alpha+i\beta)} - e^{a(\alpha+i\beta)}}{\alpha + i\beta}$$

(α 及 β 为常数).

$$\begin{aligned} \text{证 } & \int_a^b e^{(\alpha+i\beta)x} dx \\ &= \int_a^b e^{\alpha x} \cos \beta x dx + i \int_a^b e^{\alpha x} \sin \beta x dx \\ &= \frac{e^{\alpha x} [\alpha \cos \beta x + \beta \sin \beta x + i(\alpha \sin \beta x - \beta \cos \beta x)]}{\alpha^2 + \beta^2} \Big|_a^b \\ &= \frac{e^{\alpha x} (\alpha - i\beta)(\cos \beta x + i \sin \beta x)}{(\alpha + i\beta)(\alpha - i\beta)} \Big|_a^b \\ &= \frac{e^{(\alpha+i\beta)x}}{\alpha + i\beta} \Big|_a^b = \frac{e^{(\alpha+i\beta)b} - e^{(\alpha+i\beta)a}}{\alpha + i\beta}. \end{aligned}$$

利用尤拉氏公式:

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}),$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}),$$

计算下列积分(m 及 n 为正整数):

2290. $\int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx.$

解: 方法一: 记

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx,$$

易见 $I_0 = \frac{1}{4}I$, 其中

$$I = \int_0^{2\pi} \sin^{2m} x \cos^{2n} x dx.$$

利用尤拉公式, 有

$$\begin{aligned} \sin^{2m} x \cos^{2n} x &= \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^{2m} \left(\frac{e^{ix} + e^{-ix}}{2} \right)^{2n} \\ &= \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} (-1)^k C_{2m}^k e^{2(m-k)ix} \sum_{l=0}^{2n} C_{2n}^l e^{2(n-l)ix}, \\ &= \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} \sum_{l=0}^{2n} (-1)^k C_{2m}^k C_{2n}^l e^{2(m+n-k-l)ix} \\ &= \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} \sum_{l=0}^{2n} (-1)^k C_{2m}^k C_{2n}^l [\cos 2(m+n-k-l)x \\ &\quad - k-l)x + i \sin 2(m+n-k-l)x], \end{aligned}$$

今不妨设 $m \leq n^*$, 作积分计算, 则有

$$\begin{aligned} I &= \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} \sum_{l=0}^{2n} (-1)^k C_{2m}^k C_{2n}^l \left(\int_0^{2\pi} \cos(m+n \right. \\ &\quad \left. - k-l)2x dx + i \int_0^{2\pi} \sin(m+n - k-l)2x dx \right) \end{aligned}$$

* 若 $m > n$ 作代换 $x = \frac{\pi}{2} - u$ 即得.

$$\begin{aligned}
&= \frac{(-1)^m \pi}{2^{2m+2n-1}} \sum_{\substack{k+l=m+n \\ 0 \leq k \leq 2m \\ 0 \leq l \leq 2n}} (-1)^k C_{2m}^k C_{2n}^l \\
&= \frac{(-1)^m \pi}{2^{2m+2n-1}} \sum_{k=0}^{2m} (-1)^k C_{2m}^k C_{2n}^{m+n-k}
\end{aligned}$$

经计算,可以验证有*

$$\begin{aligned}
&(-1)^m \sum_{k=0}^{2m} (-1)^k C_{2m}^k C_{2n}^{m+n-k} \\
&= \frac{(2m)!(2n)!}{m!n!(m+n)!}
\end{aligned}$$

于是得

$$I_0 = \frac{1}{4} I = \frac{\pi(2m)!(2n)!}{2^{2m+2n+1} m!n!(m+n)!}.$$

方法二:

令 $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx$. 显然

$$I_{m,0} = \int_0^{\frac{\pi}{2}} \sin^{2m} x dx = \frac{(2m-1)!!}{(2m)!!} \cdot \frac{\pi}{2},$$

$$\begin{aligned}
I_{m,n} &= \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n-1} x d\sin x \\
&= \frac{1}{2m+1} \int_0^{\frac{\pi}{2}} \cos^{2n-1} x d\sin^{2m+1} x \\
&= \frac{1}{2m+1} \cos^{2n-1} x \sin^{2m+1} x \Big|_0^{\frac{\pi}{2}}
\end{aligned}$$

* 用 $C_m^k C_{2n}^{m+n-k} = C_{2n}^m C_{2n}^n (C_{m+n}^n)^{-2} C_{m+n}^k C_{m+n}^{2m-k}$, 以及由恒等式 $(1-x)^{m+n}(1+x)^{m+n} = (1-x^2)^{m+n}$ 展开, 取 x^{2m} 的系数的关系式 $\sum_{k=0}^{2m} (-1)^k C_{m+n}^k C_{m+n}^{2m-k} = (-1)^m C_{m+n}^m$ 可以验证.

$$\begin{aligned}
&= \frac{1}{2m+1} \int_0^{\frac{\pi}{2}} \sin^{2m+1} x d\cos^{2n-1} x \\
&= \frac{2n-1}{2m+1} \int_0^{\frac{\pi}{2}} \sin^{2m+2} x \cos^{2n-2} x dx \\
&= \frac{2n-1}{2m+1} \int_0^{\frac{\pi}{2}} \sin^{2m} x (1 - \cos^2 x) \cos^{2(n-1)} x dx \\
&= \frac{2n-1}{2m+1} I_{m, n-1} - \frac{2n-1}{2m+1} I_{m, n},
\end{aligned}$$

整理后得

$$I_{m, n} = \frac{2n-1}{2(m+n)} I_{m, n-1}.$$

由此不难得到

$$\begin{aligned}
I_{m, n} &= \frac{(2n-1)!!}{2^n(m+n)(m+n-1)\cdots(m+1)} I_{m, 0} \\
&= \frac{(2n-1)!!m!}{2^n(m+n)!} \cdot \frac{(2m-1)!!}{(2m)!!} \cdot \frac{\pi}{2} \\
&= \frac{\pi(2n-1)!!(2m-1)!!}{2^{m+n+1}(m+n)!} \\
&= \frac{\pi(2m)!(2n)!}{2^{2m+2n+1}m!n!(m+n)!}.
\end{aligned}$$

2291. $\int_0^\pi \frac{\sin nx}{\sin x} dx.$

解 设 $u = \frac{\sin nx}{\sin x}$, 利用尤拉公式得

$$u = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}}.$$

当 $n = 2k$ 时,

$$\begin{aligned}
u &= (e^{ikx} + e^{-ikx}) \cdot (e^{i(k-1)x} + e^{i(k-3)x} + \cdots \\
&\quad + e^{-i(k-3)x} + e^{-i(k-1)x}) \\
&= e^{(2k-1)ix} + e^{(2k-3)ix} + \cdots + e^{ix} + e^{-ix}
\end{aligned}$$

$$+ \cdots + e^{-(2k-1)ix} \\ = 2[\cos(2k-1)x + \cos(2k-3)x + \cdots + \cos x],$$

于是,

$$\int_0^\pi u dx = 2 \left[\frac{\sin(2k-1)x}{2k-1} + \frac{\sin(2k-3)x}{2k-3} \right. \\ \left. + \cdots + \sin x \right] \Big|_0^\pi = 0.$$

当 $n = 2k + 1$ 时, 同上得

$$u = 2[\cos 2kx + \cos 2(k-1)x + \cdots + \cos 2x] + 1,$$

于是,

$$\int_0^\pi u dx = \pi.$$

最后得到

$$\int_0^\pi \frac{\sin nx}{\sin x} dx = \begin{cases} 0, n \text{ 为偶数;} \\ \pi, n \text{ 为奇数.} \end{cases}$$

2292. $\int_0^\pi \frac{\cos(2n+1)x}{\cos x} dx.$

解 $\frac{\cos(2n+1)x}{\cos x} = \frac{e^{i(2n+1)x} + e^{-i(2n+1)x}}{e^{ix} + e^{-ix}}$ \\ $= e^{2nix} - e^{2(n-1)ix} + \cdots + (-1)^n + \cdots + e^{-2nix}$ \\ $= 2[\cos 2nx - \cos 2(n-1)x + \cdots$ \\ $+ (-1)^{n-1} \cos 2x] + (-1)^n.$

于是,

$$\int_0^\pi \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \pi.$$

2293. $\int_0^\pi \cos^n x \cos nx dx$

解 $\cos^n x \cos nx$

$$\begin{aligned}
&= \frac{1}{2^n + 1} (e^{ix} + e^{-ix})^n (e^{inx} + e^{-inx}) \\
&= \frac{1}{2^n} [\cos 2nx + c_n^1 \cos 2(n-1)x + \cdots \\
&\quad + c_n^{n-1} \cos 2x + 1].
\end{aligned}$$

于是,

$$\int_0^\pi \cos^n x \cos nx dx = \frac{\pi}{2^n}.$$

2294. $\int_0^\pi \sin^n x \sin nx dx.$

解 方法一:

$$\begin{aligned}
&\int_0^\pi \sin^n x \sin nx dx \\
&= \frac{1}{(2i)^{n+1}} \int_0^\pi \left[\sum_{k=0}^n (-1)^k c_n^k e^{i(n-2k)x} (e^{ix} - e^{-ix}) \right] dx \\
&= \frac{1}{(2i)^{n+1}} \left[\sum_{k=0}^n (-1)^k c_n^k \int_0^\pi e^{i(2n-2k)x} dx - \sum_{k=0}^n (-1)^k c_n^k \int_0^\pi e^{-i2kx} dx \right] \\
&= \frac{1}{2^{n+1} i^{n+1}} [(-1)^n c_n^n \cdot \pi - (-1)^0 c_n^0 \pi] \\
&= \begin{cases} 0, n \text{ 为偶数;} \\ \frac{\pi}{2^n} \cdot (-1)^{\frac{n+1}{2}+1}, n \text{ 为奇数.} \end{cases}
\end{aligned}$$

由于

$$\sin \frac{n\pi}{2} = \begin{cases} 0, n \text{ 为偶数,} \\ (-1)^{\frac{n+1}{2}+1}, n \text{ 为奇数,} \end{cases}$$

于是

$$\int_0^{\pi} \sin^n x \sin nx dx = \frac{\pi}{2^n} \sin \frac{n\pi}{2}.$$

方法二:

设 $x = \frac{\pi}{2} - t$, 则

$$\begin{aligned} & \int_0^{\pi} \sin^n x \sin nx dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \sin \left(\frac{n\pi}{2} - nt \right) dt \\ &= \sin \frac{n\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \cos nt dt \\ &\quad - \cos \frac{n\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \sin nt dt \\ &= \sin \frac{n\pi}{2} \int_0^{\pi} \cos^n x \cos nx dx \\ &= \frac{\pi}{2^n} \sin \frac{n\pi}{2}. \end{aligned}$$

求下列积分(n 为自然数):

2295. $\int_0^{\pi} \sin^{n-1} x \cos(n+1)x dx.$

$$\begin{aligned} \text{解} \quad & \int_0^{\pi} \sin^{n-1} x \cos(n+1)x dx \\ &= \int_0^{\pi} \sin^{n-1} x (\cos nx \cos x - \sin nx \sin x) dx \\ &= \int_0^{\pi} \sin^{n-1} x \cos nx d(\sin x) - \int_0^{\pi} \sin^n x \sin nx dx \\ &= \frac{\sin^n x \cos nx}{n} \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin^n x d(\cos nx) \\ &\quad - \int_0^{\pi} \sin^n x \sin nx dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{2n}} \left\{ c_{2n}^n \cdot \int_0^{2\pi} e^{-ax} dx + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
&\quad \left. \cdot \int_0^{2\pi} e^{-ax} \cos 2(n-k)x dx \right\} \\
&= \frac{1}{2^{2n}} \left\{ -\frac{1}{a} c_{2n}^n e^{-ax} \Big|_0^{2\pi} + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
&\quad \left. \cdot \frac{(2n-2k) \cdot \sin 2(n-k)x - a \cos 2(n-k)x}{a^2 + (2n-2k)^2} e^{-ax} \Big|_0^{2\pi} \right\} \\
&= \frac{1}{2^{2n}} \left\{ -\frac{1}{a} c_{2n}^n \cdot (e^{-2\pi a} - 1) - a(e^{-2\pi a} - 1) \right. \\
&\quad \left. \cdot \sum_{k=0}^{n-1} \frac{2c_{2n}^k}{a^2 + (2n-2k)^2} \right\} \\
&= \frac{1 - e^{-2\pi a}}{2^{2n} \cdot a} \left\{ c_{2n}^n + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
&\quad \left. \cdot \frac{a^2}{a^2 + (2n-2k)^2} \right\},
\end{aligned}$$

即

$$\begin{aligned}
&\int_0^{2\pi} e^{-ax} \cos^{2n} x dx \\
&= \frac{1 - e^{-2\pi a}}{2^{2n} \cdot a} \cdot \left\{ c_{2n}^n + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
&\quad \left. \cdot \frac{a^2}{a^2 + (2n-2k)^2} \right\}
\end{aligned}$$

方法二:

由于

$$\begin{aligned}
&\int_0^{2\pi} e^{(\alpha+ik)x} dx = \frac{e^{(\alpha+ik)x}}{\alpha+ik} \Big|_0^{2\pi} \\
&= \frac{e^{2\pi\alpha} - 1}{\alpha+ik}
\end{aligned}$$

$$= \frac{(e^{2\pi a} - 1)(a - ik)}{a^2 + k^2},$$

取实部, 得

$$\operatorname{Re} \int_0^{2\pi} e^{(a+ik)x} dx = \frac{a(e^{2\pi a} - 1)}{a^2 + k^2}$$

于是,

$$\begin{aligned} & \int_0^{2\pi} e^{-ax} \cos^{2n} x dx \\ &= \int_0^{2\pi} e^{-ax} \left(\frac{e^{ix} + e^{-ix}}{2} \right)^{2n} dx \\ &= \frac{1}{2^{2n}} \int_0^{2\pi} e^{-ax} \left(\sum_{k=0}^{2n} c_{2n}^k e^{i(2n-2k)x} \right) dx \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} c_{2n}^k \int_0^{2\pi} e^{-a+i(2n-2k)x} dx \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} c_{2n}^k \frac{e^{-2\pi a} - 1}{-a + i(2n - 2k)} \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} c_{2n}^k \frac{a(1 - e^{-2\pi a})}{a^2 + (2n - 2k)^2} \\ &= \frac{1 - e^{-2\pi a}}{2^{2n} \cdot a} \left[c_{2n}^n + 2 \sum_{k=0}^{n-1} c_{2n}^k \frac{a^2}{a^2 + (2n - 2k)^2} \right]. \end{aligned}$$

2298. $\int_0^{\frac{\pi}{2}} \ln \cos x \cdot \cos 2nx dx.$

解 利用分部积分得

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \ln \cos x \cdot \cos 2nx dx \\ &= \frac{1}{2n} \sin 2nx \cdot \ln \cos x \Big|_0^{\frac{\pi}{2}} \\ &+ \frac{1}{2n} \int_0^{\frac{\pi}{2}} \frac{\sin 2nx \cdot \sin x}{\cos x} dx \end{aligned}$$

$$= 0^{+1} + \frac{1}{4n} \int_0^{\frac{\pi}{2}} \frac{\cos(2n-1)x}{\cos x} dx \\ - \frac{1}{4n} \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} dx.$$

对于上述等式右端的第二项和第三项的被积函数有下列等式:

$$\frac{\cos(2n-1)x}{\cos x} = \frac{e^{i(2n-1)x} + e^{-i(2n-1)x}}{e^{ix} + e^{-ix}} \\ = 2[\cos 2(n-1)x - \cos 2(n-2)x + \dots \\ + (-1)^{n-2} \cos 2x] + (-1)^{n-1}, \\ \frac{\cos(2n+1)x}{\cos x} \\ = 2[\cos 2nx - \cos 2(n-1)x + \dots \\ + (-1)^{n-1} \cos 2x] + (-1)^n.$$

由于积分

$$\int_0^{\frac{\pi}{2}} \cos 2kx dx \quad (k \text{ 为任意的正整数})$$

的值恒等于零, 所以积分

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2n-1)x}{\cos x} dx \quad \text{及} \quad \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} dx$$

分别等于 $(-1)^{n-1} \frac{\pi}{2}$ 及 $(-1)^n \frac{\pi}{2}$.

这样, 我们得到

$$\int_0^{\frac{\pi}{2}} \ln \cos x \cdot \cos 2nxdx \\ = \frac{1}{4n} \left[(-1)^{n-1} \frac{\pi}{2} - (-1)^n \frac{\pi}{2} \right] \\ = \frac{\pi}{4n} (-1)^{n-1}.$$

*) 在 $x = 0$ 处, $\sin 2nx \cdot \ln \cos x = 0$; 而在 $x = \frac{\pi}{2}$ 处, 为“ $0 \cdot \infty$ ”型, 采用洛比塔法则定值:

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}-0} \sin 2nx \cdot \ln \cos x &= \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\ln \cos x}{\frac{1}{\sin 2nx}} \\ &= \frac{1}{2n} \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\sin x \cdot \sin^2 2nx}{\cos x \cdot \cos 2nx} \\ &= \frac{1}{2n} \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\cos x \cdot \sin^2 2nx + 4n \sin x \sin 2nx \cos 2nx}{-\sin x \cos 2nx - 2n \cos x \sin 2nx} \\ &= 0. \end{aligned}$$

2299. 利用多次的部分积分法, 计算尤拉氏积分:

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx,$$

式中 m 及 n 为正整数.

$$\begin{aligned} \text{解 } B(m, n) &= \frac{1}{m} x^m (1-x)^{n-1} \Big|_0^1 \\ &\quad + \frac{n-1}{m} \int_0^1 x^m (1-x)^{n-2} dx \\ &= \frac{n-1}{m} B(m+1, n-1). \end{aligned}$$

继续利用部分积分法, 可得

$$\begin{aligned} B(m, n) &= \frac{(n-1)(n-2)\cdots 2 \cdot 1}{m(m+1)\cdots(m+n-2)} \int_0^1 x^{m+n-2} dx \\ &= \frac{(n-1)!(m-1)!}{(m+n-2)!} \\ &\quad \cdot \frac{1}{m+n-1} x^{m+n-1} \Big|_0^1 \\ &= \frac{(n-1)!(m-1)!}{(m+n-1)!}. \end{aligned}$$

2300. 勒让德多项式 $P_n(x)$ 被下面公式来定义:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n] (n = 0, 1, 2, \dots).$$

证明

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0, & \text{若 } m \neq n, \\ \frac{2}{2n+1}, & \text{若 } m = n. \end{cases}$$

证 当 $m \neq n$ 时, 不失一般性, 设 $n < m$. 由于 $P_m(x)$ 为一 m 次的多项式, 我们记

$$P_m(x) = R^{(m)}(x),$$

其中 $R(x) = \frac{1}{2^n n!} (x^2 - 1)^n$.

利用多次部分积分法得

$$\begin{aligned} & \int_{-1}^1 P_m(x) P_n(x) dx \\ &= [P_n(x) R^{(m-1)}(x) - P_n'(x) R^{(m-2)}(x) + \dots \\ & \quad + (-1)^{m-1} P_n^{(m-1)}(x) R(x)] \Big|_{-1}^1 \\ & \quad + (-1)^m \int_{-1}^1 R(x) P_n^{(m)}(x) dx = 0. \end{aligned}$$

当 $m = n$ 时,

$$\begin{aligned} & \int_{-1}^1 P_m(x) P_n(x) dx \\ &= \frac{1}{2^{2n} (n!)^2} \int_{-1}^1 \left[\frac{d^n (x^2 - 1)^n}{dx^n} \right]^2 dx, \end{aligned}$$

设 $u = \frac{d^n}{dx^n} [(x^2 - 1)^n]$, $v = (x^2 - 1)^n$, 则

$$\int_{-1}^1 P_n^2(x) dx$$

$$\begin{aligned}
&= \frac{1}{2^{2n}(n!)^2} [uv^{(n-1)} - u'v^{(n-2)} + \dots \\
&\quad + (-1)^{n-1}u^{(n-1)}v] \Big|_{-1}^1 + (-1)^n \\
&\quad \cdot \frac{1}{2^{2n}(n!)^2} \int_{-1}^1 vu^{(n)}dx \\
&= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^1 (x^2-1)^n \cdot \frac{d^{2n}}{dx^{2n}} [(x^2-1)^n] dx \\
&= \frac{(2n)!}{2^{2n-1}(n!)^2} \int_0^1 (1-x^2)^n dx \\
&= \frac{(2n)!}{2^{2n-1}(n!)^2} \int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt \\
&= \frac{(2n)!}{2^{2n-1}(n!)^2} \cdot \frac{(2n)!!}{(2n+1)!!} = \frac{2}{2n+1}.
\end{aligned}$$

*) 设 $x = \sin t$.

* *) 利用 2282 题的结果.

2301. 设函数 $f(x)$ 在 $[a, b]$ 上可积分, 函数 $F(x)$ 在 $[a, b]$ 内除了有限个点 $c_i (i = 1, \dots, p)$ 及点 a 与 b 外皆有 $F'(x) = f(x)$, 而在这除去的有限个点上 $F(x)$ 有第一类的间断点(广义原函数). 证明

$$\begin{aligned}
\int_a^b f(x) dx &= F(b-0) - F(a+0) \\
&\quad - \sum_{i=1}^p [F(c_i+0) - F(c_i-0)].
\end{aligned}$$

证 为确定起见, 设 $a < c_1 < c_2 < \dots < c_p < b$, 并记 $a = c_0, b = c_{p+1}$. 由于 $f(x)$ 在 $[a, b]$ 上可积, 故

$$\int_a^b f(x) dx = \lim_{\eta \rightarrow 0+} \sum_{i=0}^p \int_{c_i+\eta}^{c_{i+1}-\eta} f(x) dx.$$

显然, 在 $[c_i + \eta, c_{i+1} - \eta]$ 上 $F'(x) = f(x)$, 从而可应

用牛顿—莱布尼兹公式,得

$$\int_{c_i+\eta}^{c_{i+1}-\eta} f(x)dx = F(c_{i+1}-\eta) - F(c_i+\eta),$$

由此可知

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{\eta \rightarrow 0+} \sum_{i=0}^p [F(c_{i+1}-\eta) - F(c_i+\eta)] \\ &= \sum_{i=0}^p [F(c_{i+1}-0) - F(c_i+0)] \\ &= F(b-0) - F(a+0) - \sum_{i=1}^p [F(c_i+0) \\ &\quad - F(c_i-0)].\end{aligned}$$

2302. 设函数 $f(x)$ 在闭区间 $[a, b]$ 上可积分, 而

$$F(x) = C + \int_a^x f(\xi)d\xi$$

为 $f(x)$ 的不定积分, 证明函数 $F(x)$ 连续且在函数 $f(x)$ 连续的一切点处有等式

$$F'(x) = f(x)$$

成立, 问在函数 $f(x)$ 不连续点处函数 $F(x)$ 的导函数为何?

解 由于 $f(x)$ 在 $[a, b]$ 上可积, 故必有界; $|f(x)| \leq M$ ($a \leq x \leq b$). 因此, 对任何 $x \in [a, b]$, 有

$$\begin{aligned}&|F(x+\Delta x) - F(x)| \\ &= \left| \int_x^{x+\Delta x} f(\xi)d\xi \right| \leq M \cdot |\Delta x| \rightarrow 0 \text{ (当 } \Delta x \rightarrow 0 \text{ 时)}.\end{aligned}$$

由此可知 $F(x)$ 在 $[a, b]$ 上连续.

现设 $f(\xi)$ 在点 $\xi = x$ 处连续. 于是, 任给 $\epsilon > 0$, 存在 $\delta > 0$, 使当 $|\xi - x| < \delta$ 时, 恒有 $|f(\xi) - f(x)| <$

ϵ . 于是, 当 $0 < |\Delta x| < \delta$ 时, 恒有

$$\begin{aligned} & \left| \frac{F(x + \Delta x) - F(x)}{\Delta x} - f(x) \right| \\ &= \left| \frac{1}{\Delta x} \int_x^{x+\Delta x} [f(\xi) - f(x)] d\xi \right| \\ &< \frac{1}{|\Delta x|} \epsilon \cdot |\Delta x| = \epsilon, \end{aligned}$$

故 $F'(x)$ 存在, 且

$$F'(x) = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x).$$

而在不连续点处 $F'(x)$ 可能存在也可能不存在.

例如, 设

$$f(x) = \begin{cases} 1, & \text{当 } x = \frac{1}{n}, \\ 0, & \text{当 } x \neq \frac{1}{n}, \end{cases} \quad (n = 1, 2, 3, \dots).$$

则 $f(x)$ 在 $[0, 1]$ 的可积性可仿 2194 题证明, 而且显然有

$$\int_0^x f(t) dt = 0 \quad (0 \leq x \leq 1).$$

然而在点 $x = \frac{1}{n}$ 处, $F(x) = C$ 的导函数 $F'(x) = 0$ 是存在的.

但函数 $f(x) = \operatorname{sgn} x$, 它在 $(-1, 1)$ 上是可积的, 且

$$\int_0^x f(x) dx = |x|,$$

然而在点 $x = 0$ 处, $F(x) = |x| + C$ 的导函数 $F'(x)$ 不存在.

求下列有界非连续函数的不定积分:

2303. $\int \operatorname{sgn} x dx.$

解 $\int \operatorname{sgn} x dx = \int_0^x \operatorname{sgn} x dx + C = |x| + C.$

2304. $\int \operatorname{sgn}(\sin x) dx.$

解 由于 $\operatorname{sgn}(\sin x)$ 在任何有限区间上都可积, 故其原函数 $F(x) = \int_0^x \operatorname{sgn}(\sin t) dt$ 是 $(-\infty, +\infty)$ 上的连续函数. 对任何 x , 必存在唯一的整数 k 使 $k\pi \leq x < (k+1)\pi$. 于是

$$\begin{aligned} F(x) &= \int_0^x \operatorname{sgn}(\sin t) dt \\ &= \int_0^{k\pi + \frac{\pi}{2}} \operatorname{sgn}(\sin t) dt + \int_{k\pi + \frac{\pi}{2}}^x \operatorname{sgn}(\sin t) dt \\ &= \frac{\pi}{2} + \int_{k\pi + \frac{\pi}{2}}^x \frac{\sin t}{\sqrt{1 - \cos^2 t}} dt \\ &= \frac{\pi}{2} + \arccos(\cos t) \Big|_{k\pi + \frac{\pi}{2}}^x \\ &= \frac{\pi}{2} + \arccos(\cos x) - \frac{\pi}{2} \\ &= \arccos(\cos x). \end{aligned}$$

故

$$\begin{aligned} \int \operatorname{sgn}(\sin x) dx &= \arccos(\cos x) + C \\ (-\infty < x < +\infty). \end{aligned}$$

2305. $\int [x] dx (x \geq 0).$

解 $\int [x] dx = C + \int_0^x [x] dx$

$$\begin{aligned}
&= \int_0^x \operatorname{sgn}(\sin \pi x) dx + C \\
&= \frac{1}{\pi} \arccos(\cos \pi x) \Big|_0^{x^{**}} + C \\
&= \frac{1}{\pi} \arccos(\cos \pi x) + C.
\end{aligned}$$

*) 利用 2304 题的结果.

2308. $\int_0^x f(x) dx$, 其中 $f(x) = \begin{cases} 1, & \text{若 } |x| < l, \\ 0, & \text{若 } |x| > l. \end{cases}$

解
$$\begin{aligned}
\int_0^x f(x) dx &= \int_0^l f(x) dx + \int_l^x f(x) dx \\
&= \int_0^l 1 \cdot dx + \int_0^x 0 dx = l \quad (x \geqslant l), \\
\int_0^x f(x) dx &= \int_0^x 1 \cdot dx = x \quad (|x| < l), \\
\int_0^x f(x) dx \\
&= - \int_x^{-l} f(x) dx - \int_{-l}^0 f(x) dx = -l \quad (x \leqslant -l).
\end{aligned}$$

合并得

$$\int_0^x f(x) dx = \frac{1}{2} (|l+x| - |l-x|).$$

计算下列有界非连续函数的定积分.

2309. $\int_0^3 \operatorname{sgn}(x - x^3) dx$.

解 $\operatorname{sgn}(x - x^3) = \begin{cases} 1, & \text{当 } x \in (0, 1) \text{ 时,} \\ -1, & \text{当 } x \in (1, 3) \text{ 时.} \end{cases}$

于是,

$$\int_0^3 \operatorname{sgn}(x - x^3) dx = \int_0^1 dx - \int_1^3 dx = -1.$$

$$2310. \int_0^2 [e^x] dx.$$

$$\begin{aligned} \text{解} \quad & \int_0^2 [e^x] dx \\ &= \int_0^{\ln 2} 1 dx + \int_{\ln 2}^{\ln 3} 2 \cdot dx + \int_{\ln 3}^{\ln 4} 3 \cdot dx \\ &\quad + \cdots + \int_{\ln 7}^2 7 \cdot dx \\ &= \ln 2 + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + \cdots \\ &\quad + 7(-\ln 7 + 2) \\ &= 14 - (\ln 2 + \ln 3 + \ln 4 + \cdots + \ln 7) \\ &= 14 - \ln 7!. \end{aligned}$$

$$2311. \int_0^6 [x] \sin \frac{\pi x}{6} dx.$$

$$\begin{aligned} \text{解} \quad & \int_0^6 [x] \sin \frac{\pi x}{6} dx \\ &= \int_1^2 \sin \frac{\pi x}{6} dx + \int_2^3 2 \sin \frac{\pi x}{6} dx + \cdots \\ &\quad + \int_5^6 5 \sin \frac{\pi x}{6} dx \\ &= \frac{30}{\pi}. \end{aligned}$$

$$2312. \int_0^{\pi} x \operatorname{sgn}(\cos x) dx.$$

$$\begin{aligned} \text{解} \quad & \int_0^{\pi} x \operatorname{sgn}(\cos x) dx \\ &= \int_0^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\pi} (-x) dx = -\frac{\pi^2}{4}. \end{aligned}$$

$$2313. \int_1^{n+1} \ln [x] dx, \text{ 其中 } n \text{ 为自然数.}$$

$$\text{解} \quad \int_1^{n+1} \ln [x] dx$$

$$\begin{aligned}
&= \int_2^3 \ln 2 dx + \int_3^4 \ln 3 dx + \cdots + \int_n^{n+1} \ln n dx \\
&= \ln n!.
\end{aligned}$$

2314. $\int_0^1 \operatorname{sgn}[\sin(\ln x)] dx.$

解
$$\begin{aligned}
&\int_0^1 \operatorname{sgn}[\sin(\ln x)] dx \\
&= \int_{e^{-\pi}}^1 (-1) dx + \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k+1} \int_{e^{-(k+1)\pi}}^{e^{-k\pi}} dx \\
&= -1 + 2e^{-\pi} \cdot \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} e^{-(k-1)\pi} \\
&= -1 + \frac{2e^{-\pi}}{1 + e^{-\pi}} = \frac{e^{-\pi} - 1}{e^{-\pi} + 1} \\
&= -th \frac{\pi}{2}.
\end{aligned}$$

2315. 求 $\int_E |\cos x| \sqrt{\sin x} dx$, 其中 E 为闭区间 $(0, 4\pi)$ 中使被积分式有意义的一切值所成之集合.

解
$$\begin{aligned}
&\int_E |\cos x| \sqrt{\sin x} dx \\
&= \int_0^{\pi} |\cos x| \sqrt{\sin x} dx + \int_{2\pi}^{3\pi} |\cos x| \sqrt{\sin x} dx \\
&= \int_0^{\frac{\pi}{2}} \cos x \sqrt{\sin x} dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \sqrt{\sin x} dx \\
&\quad + \int_{2\pi}^{\frac{5\pi}{2}} \cos x \sqrt{\sin x} dx + \int_{\frac{5\pi}{2}}^{3\pi} (-\cos x) \sqrt{\sin x} dx \\
&= 4 \int_0^{\frac{\pi}{2}} \cos x \sqrt{\sin x} dx \\
&= \frac{8}{3} (\sin x)^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} = \frac{8}{3}.
\end{aligned}$$

§ 3. 中值定理

1° 函数的平均值 数

$$M[f] = \frac{1}{b-a} \int_a^b f(x) dx$$

称为函数 $f(x)$ 在区间 $[a, b]$ 上的平均值.

若函数 $f(x)$ 在 $[a, b]$ 上连续, 则可求得一点 $c \in (a, b)$ 适合

$$M[f] = f(c).$$

2° 第一中值定理 若: (1) 函数 $f(x)$ 和 $\varphi(x)$ 于闭区间 $[a, b]$ 上有界并可积分; (2) 当 $a < x < b$ 时, 函数 $\varphi(x)$ 不变号, 则

$$\int_a^b f(x)\varphi(x)dx = \mu \int_a^b \varphi(x)dx,$$

式中 $m \leq \mu \leq M$ 及 $M = \sup_{a < x < b} f(x)$, $m = \inf_{a < x < b} f(x)$; (3) 除此而外, 若函数 $f(x)$ 于闭区间 $[a, b]$ 上连续, 则 $\mu = f(c)$, 其中 $a \leq c \leq b$ (编者注: 可以证明, c 可取值使 $a < c < b$).

3° 第二中值定理 若: (1) 函数 $f(x)$ 和 $\varphi(x)$ 于闭区间 $[a, b]$ 上有界并可积分; (2) 当 $a < x < b$ 时, 函数 $\varphi(x)$ 是单调的, 则

$$\begin{aligned} & \int_a^b f(x)\varphi(x)dx \\ &= \varphi(a+0) \int_a^{\xi} f(x)dx + \varphi(b-0) \int_{\xi}^b f(x)dx, \end{aligned}$$

式中 $a \leq \xi \leq b$; (3) 除此而外, 若函数 $\varphi(x)$ 单调下降 (广义的) 且不为负, 则

$$\int_a^b f(x)\varphi(x)dx = \varphi(a+0) \int_a^{\xi} f(x)dx \quad (a \leq \xi \leq b);$$

(3') 若函数 $\varphi(x)$ 单调上升 (广义的) 且不为负, 则

$$\int_a^b f(x)\varphi(x)dx = \varphi(b-0) \int_{\xi}^b f(x)dx \quad (a \leq \xi \leq b).$$

2316. 确定下列定积分的符号:

$$\begin{array}{ll} \text{(a)} \int_0^{2\pi} x \sin x dx; & \text{(6)} \int_0^{2\pi} \frac{\sin x}{x} dx; \\ \text{(B)} \int_{-2}^2 x^3 2^x dx; & \text{(r)} \int_{\frac{1}{2}}^1 x^2 \ln x dx. \end{array}$$

解

$$\begin{aligned} \text{(a)} \int_0^{2\pi} x \sin x dx &= \int_0^{\pi} x \sin x dx + \int_{\pi}^{2\pi} x \sin x dx \\ &= \int_0^{\pi} x \sin x dx - \int_0^{\pi} (t + \pi) \sin t dt \\ &= -\pi \int_0^{\pi} \sin x dx < 0. \end{aligned}$$

(6) 由第一中值定理知

$$\begin{aligned} \int_0^{2\pi} \frac{\sin x}{x} dx &= \int_0^{\pi} \frac{\sin x}{x} dx + \int_{\pi}^{2\pi} \frac{\sin x}{x} dx \\ &= \int_0^{\pi} \frac{\sin x}{x} dx - \int_0^{\pi} \frac{\sin t}{t + \pi} dt \\ &= \pi \int_0^{\pi} \frac{\sin x}{x(x + \pi)} dx \\ &= \frac{\pi^2 \sin c}{c(c + \pi)} > 0, \end{aligned}$$

其中 $0 < c < \pi$.

(B) 由第一中值定理知

$$\begin{aligned} \int_{-2}^2 x^3 e^x dx &= \int_{-2}^0 x^3 e^x dx + \int_0^2 x^3 e^x dx \\ &= \int_2^0 t^3 e^{-t} dt + \int_0^2 x^3 e^x dx \end{aligned}$$

$$= \int_0^2 x^3 (e^x - e^{-x}) dx = 2c^3 (e^c - e^{-c}) > 0,$$

其中 $0 < c < 2$.

$$\begin{aligned} \text{(r)} \quad & \int_{\frac{1}{2}}^1 x^2 \ln x dx \\ &= \frac{1}{2} c^2 \ln c < 0 \quad \left(\text{其中 } \frac{1}{2} < c < 1 \right) \end{aligned}$$

2317. 于下列各题中确定那个积分较大:

$$\text{(a)} \quad \int_0^{\frac{\pi}{2}} \sin^{10} x dx \text{ 或 } \int_0^{\frac{\pi}{2}} \sin^2 x dx?$$

$$\text{(6)} \quad \int_0^1 e^{-x} dx \text{ 或 } \int_0^1 e^{-x^2} dx?$$

$$\text{(B)} \quad \int_0^{\pi} e^{-x^2} \cos^2 x dx \text{ 或 } \int_{\pi}^{2\pi} e^{-x^2} \cos^2 x dx?$$

解 (a) 当 $x \in \left(0, \frac{\pi}{2}\right)$ 时, $0 < \sin x < 1$ 从而
 $0 < \sin^{10} x < \sin^2 x,$

于是

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx < \int_0^{\frac{\pi}{2}} \sin^2 x dx.$$

(6) 当 $0 < x < 1$ 时, $x > x^2$, 从而

$$e^{-x} < e^{-x^2},$$

于是

$$\int_0^1 e^{-x} dx < \int_0^1 e^{-x^2} dx.$$

$$\begin{aligned} \text{(B)} \quad & \int_{\pi}^{2\pi} e^{-x^2} \cos^2 x dx \\ &= \int_0^{\pi} e^{-(\pi+x)^2} \cos^2 x dx < \int_0^{\pi} e^{-x^2} \cos^2 x dx. \end{aligned}$$

2318. 求下列已知函数在所给区间内的平均值:

(a) $f(x) = x^2$ 在 $[0, 1]$ 上;

(b) $f(x) = \sqrt{x}$ 在 $[0, 100]$ 上;

(c) $f(x) = 10 + 2\sin x + 3\cos x$ 在 $[0, 2\pi]$ 上;

(d) $f(x) = \sin x \sin(x + \varphi)$ 在 $[0, 2\pi]$ 上.

解 (a) $M[f] = \int_0^1 x^2 dx = \frac{1}{3};$

$$(b) M[f] = \frac{1}{100} \int_0^{100} \sqrt{x} dx = 6 \frac{2}{3};$$

$$(c) M[f] = \frac{1}{2\pi} \int_0^{2\pi} (10 + 2\sin x + 3\cos x) dx \\ = 10;$$

$$(d) M[f] = \frac{1}{2\pi} \int_0^{2\pi} \sin x \cdot \sin(x + \varphi) dx \\ = \frac{1}{2} \cos \varphi.$$

2319. 求椭圆之焦径

$$r = \frac{p}{1 - \epsilon \cos \varphi} \quad (0 < \epsilon < 1)$$

之长的平均值.

解 设 $\varphi = \pi + t$, 则

$$\begin{aligned} M(r) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{p}{1 - \epsilon \cos \varphi} d\varphi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{p}{1 + \epsilon \cos t} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{p}{1 + \epsilon \cos \varphi} d\varphi \\ &= \frac{p}{2\pi} \frac{2\pi}{\sqrt{1 - \epsilon^2}} \\ &= \frac{p}{\sqrt{1 - \epsilon^2}} = b, \end{aligned}$$

其中 b 为椭圆的短半轴.

*) 利用 2213 题的结果.

2320. 求初速度为 v_0 之自由落体的速度之平均值.

解 自由落体的速度为 $v = v_0 + gt$, 从 $t = 0$ 到 $t = T$ 时间内的速度的平均值

$$\begin{aligned} M[v] &= \frac{1}{T} \int_0^T (v_0 + gt) dt = \frac{1}{2} gT + v_0 \\ &= \frac{1}{2} (v_0 + v_T). \end{aligned}$$

物理意义: 平均速度等于初速与末速之和的一半.

2321. 电流的强度依下面的规律变化

$$i = i_0 \sin\left(\frac{2\pi t}{T} + \varphi\right),$$

其中 i_0 表振幅, t 表时间, T 表周期, φ 表初相, 求电流强度之平方的平均值.

$$\begin{aligned} \text{解 } M(i^2) &= \frac{1}{T} \int_0^T i_0^2 \sin^2\left(\frac{2\pi t}{T} + \varphi\right) dt \\ &= \frac{i_0^2}{2\pi} \left[\frac{1}{2} \left(\frac{2\pi t}{T} + \varphi \right) \right. \\ &\quad \left. - \frac{1}{4} \sin 2\left(\frac{2\pi t}{T} + \varphi \right) \right] \Big|_0^T = \frac{i_0^2}{2}. \end{aligned}$$

将上式开平方, 即得电流的有效值 $\frac{i_0}{\sqrt{2}}$.

2322. 命 $\int_0^x f(t) dt = xf(\theta x)$, 求 θ , 设:

(a) $f(t) = t^n (n > -1)$; (b) $f(t) = \ln t$;

(B) $f(t) = e^t$,

$\lim_{x \rightarrow 0} \theta$ 及 $\lim_{x \rightarrow +\infty} \theta$ 等于甚么?

解 (a) $\int_0^x f(t)dt = \int_0^x t^n dt = \frac{x^{n+1}}{n+1}$, 从而

$$\frac{x^{n+1}}{n+1} = \theta^n x^{n+1}.$$

于是

$$\theta = \sqrt[n]{\frac{1}{n+1}}.$$

$$\begin{aligned} (b) \int_0^x f(t)dt &= \int_0^x \ln t dt \\ &= t(\ln t - 1) \Big|_0^x \\ &= x(\ln x - 1), \end{aligned}$$

从而

$$x(\ln x - 1) = x \ln \theta x,$$

于是

$$\theta = \frac{1}{e}.$$

$$\begin{aligned} (B) \int_0^x f(t)dt &= \int_0^x e^t dt = e^t \Big|_0^x = e^x - 1, \text{ 从而} \\ e^x - 1 &= x e^{\theta x}, \end{aligned}$$

于是

$$\theta = \frac{1}{x} \ln \frac{e^x - 1}{x}.$$

由于 $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$, 故当 $x \rightarrow 0$ 时, $\frac{1}{x} \ln \frac{e^x - 1}{x}$ 是 $\frac{0}{0}$ 型未定形. 因此

$$\begin{aligned} \lim_{x \rightarrow 0} \theta &= \lim_{x \rightarrow 0} \frac{1}{x} \ln \frac{e^x - 1}{x} \\ &= \lim_{x \rightarrow 0} \left(\frac{x}{e^x - 1} \cdot \frac{x e^x - (e^x - 1)}{x^2} \right) \end{aligned}$$

$$2324. \int_0^1 \frac{x^9}{\sqrt{1+x}} dx.$$

解 由于 $\frac{x^9}{\sqrt{2}} \leq \frac{x^9}{\sqrt{1+x}} \leq x^9 (0 \leq x \leq 1)$, 从而,

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_0^1 x^9 dx &\leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \\ &\leq \int_0^1 x^9 dx, \end{aligned}$$

即

$$\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \frac{1}{10}.$$

$$2325. \int_0^{100} \frac{e^{-x}}{x+100} dx.$$

$$\begin{aligned} \text{解 } I &= \int_0^{50} \frac{e^{-x}}{x+100} dx + \int_{50}^{100} \frac{e^{-x}}{x+100} dx \\ &= \frac{1}{100+\xi_1} \int_0^{50} e^{-x} dx + \frac{1}{100+\xi_2} \int_{50}^{100} e^{-x} dx \\ &= \frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_2}, \text{ 其中 } 0 \leq \xi_1 \leq 50, \\ &\quad 50 \leq \xi_2 \leq 100. \end{aligned}$$

显然

$$\begin{aligned} &\frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_2} \\ &\leq \frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_1} \\ &= \frac{1-e^{-100}}{100+\xi_1} < \frac{1}{100}, \\ &\frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_2} \\ &> \frac{1-e^{-50}}{100+\xi_1} \geq \frac{1-e^{-50}}{150} > \frac{1}{200}, \end{aligned}$$

故 $\frac{1}{200} < I < \frac{1}{100}$, 即 $I = 0.01 - 0.005\theta, 0 < \theta < 1$.

另外, 按中值定理, 可写

$$\begin{aligned} I &= \int_0^{100} \frac{e^{-x}}{x+100} dx = \frac{1}{\xi+100} \int_0^{100} e^{-x} dx \\ &= \frac{1}{\xi+100} \left(1 - \frac{1}{e^{100}} \right), \end{aligned}$$

其中 $0 \leq \xi \leq 100$, 如果改写 I 为

$$I = 0.01 - 0.005\theta,$$

则有

$$\theta = f(\xi) = \frac{2}{100 + \xi} \left(\xi + \frac{100}{e^{100}} \right).$$

易见导数

$$f'(\xi) = \frac{200(1 - e^{-100})}{(100 + \xi)^2} > 0,$$

$f(\xi)$ 单调上升, 故在 $[0, 100]$ 上有 $f(0) \leq f(\xi) \leq f(100)$, 也即有

$$\frac{2}{e^{100}} \leq \theta \leq 1 + \frac{1}{e^{100}}.$$

根据前面的估计 $0 < \theta < 1$, 综合起来, 便有

$$\frac{2}{e^{100}} \leq \theta < 1.$$

这个结果比原来的估计又好了一些. 如果更精确一些, 采用些近似计算方法, 还可进一步明确 θ 的数值范围. 此处从略.

2326. 证明等式

$$(a) \lim_{n \rightarrow \infty} \int_0^1 \frac{x^n}{1+x} dx = 0; \quad (b) \lim_{n \rightarrow \infty} \int_0^{\frac{\pi}{2}} \sin^n x dx = 0.$$

证 (a) $\lim_{n \rightarrow \infty} \int_0^1 \frac{x^n}{1+x} dx = \lim_{n \rightarrow \infty} \frac{1}{1+\xi_n} \int_0^1 x^n dx$
 $= \lim_{n \rightarrow \infty} \frac{1}{1+\xi_n} \cdot \frac{1}{n+1} = 0;$

(6) 任意给定 $\varepsilon > 0$, 且设 $\varepsilon < \frac{\pi}{2}$, 则

$$0 \leq \int_0^{\frac{\pi}{2}} \sin^n x dx \leq \int_0^{\frac{\pi}{2}-\varepsilon} \sin^n x dx + \varepsilon$$

$$\leq \varepsilon + \left(\frac{\pi}{2} - \varepsilon \right) \sin^n \left(\frac{\pi}{2} - \varepsilon \right).$$

当 $n \rightarrow \infty$ 时, 上述不等式的第二项趋于零, 于是

$$\lim_{n \rightarrow \infty} \int_0^{\frac{\pi}{2}} \sin^n x dx = 0.$$

2327. 设函数 $f(x)$ 在 $[a, b]$ 上连续, 而 $\varphi(x)$ 在 $[a, b]$ 上连续且在 (a, b) 上可微分, 并且

$$\varphi'(x) \geq 0 \quad \text{当} \quad a < x < b.$$

应用部分积分法及第一中值定理以证明第二中值定理.

证 设 $F(x) = \int_a^x f(t) dt$, 则

$$\begin{aligned} \int_a^b f(x) \varphi(x) dx &= \int_a^b \varphi(x) dF(x) \\ &= F(x) \varphi(x) \Big|_a^b - \int_a^b F(x) \varphi'(x) dx \\ &= F(b) \varphi(b) - F(a) \varphi(a) - F(\xi) \int_a^b \varphi'(x) dx \\ &= F(b) \varphi(b) - F(a) \varphi(a) - F(\xi) [\varphi(b) - \varphi(a)]^{**} \\ &= \varphi(b) [F(b) - F(\xi)] + \varphi(a) [F(\xi) - F(a)] \\ &= \varphi(b) \int_{\xi}^b f(x) dx + \varphi(a) \int_a^{\xi} f(x) dx. \end{aligned}$$

*) 一般数学分析中已有第二中值定理的证明, 本题限用部分积分法证明, 应加 $\varphi(x)$ 在 $[a, b]$ 上连续的条件.

利用第二中值定理, 估计积分:

$$2328. \int_{100\pi}^{200\pi} \frac{\sin x}{x} dx.$$

解 设 $f(x) = \sin x$, $\varphi(x) = \frac{1}{x}$, 则 $f(x)$ 及 $\varphi(x)$ 在 $[100\pi, 200\pi]$ 上满足第二中值定理的条件, 又 $\varphi(x) = \frac{1}{x}$ 单调下降且不为负, 于是,

$$\begin{aligned} \int_{100\pi}^{200\pi} \frac{\sin x}{x} dx &= \frac{1}{100\pi} \int_{100\pi}^{\xi} \sin x dx \\ &= \frac{1 - \cos \xi}{100\pi} = \frac{\sin^2 \frac{\xi}{2}}{50\pi} = \frac{\theta}{50\pi}, \end{aligned}$$

其中 $100\pi \leq \xi \leq 200\pi$ 及 $0 \leq \theta \leq 1$.

$$2329. \int_a^b \frac{e^{-ax}}{x} \sin x dx \quad (\alpha \geq 0; 0 < a < b).$$

解 设 $f(x) = \sin x$, $\varphi(x) = \frac{e^{-ax}}{x}$, 同上题, 有

$$\begin{aligned} \int_a^b \frac{e^{-ax}}{x} \sin x dx &= \frac{e^{-a\xi}}{a} \int_a^{\xi} \sin x dx \\ &= \frac{1}{ae^{a\xi}} (\cos a - \cos \xi) \\ &= -\frac{2}{a} e^{-a\xi} \sin \frac{a + \xi}{2} \sin \frac{a - \xi}{2} = \frac{2}{a} \theta, \end{aligned}$$

其中 $a \leq \xi \leq b$ 及 $|\theta| < 1$.

$$2330. \int_a^b \sin x^2 dx \quad (0 < a < b).$$

解 设 $x = \sqrt{t}$, 则

$$\int_a^b \sin x^2 dx = \frac{1}{2} \int_{a^2}^{b^2} \frac{\sin t}{\sqrt{t}} dt.$$

其次, 设 $f(t) = \sin t$, $\varphi(t) = (\sqrt{t})^{-1}$, 则 $\varphi(t)$ 单调下降, 且 $\varphi(t) > 0$, 于是

$$\begin{aligned} \frac{1}{2} \int_{a^2}^{b^2} \frac{\sin t}{\sqrt{t}} dt &= \frac{1}{2a} \int_{a^2}^{\xi} \sin t dt \\ &= \frac{1}{2a} (\cos a^2 - \cos \xi) \\ &= \frac{1}{a} \sin \frac{\xi + a^2}{2} \sin \frac{\xi - a^2}{2} \\ &= \frac{1}{a} \theta, \end{aligned}$$

其中 $a^2 \leq \xi \leq b^2$, $|\theta| \leq 1$. 所以

$$\int_a^b \sin x^2 dx = \frac{\theta}{a} \quad (|\theta| \leq 1).$$

2331. 设函数 $\varphi(x)$ 及 $\psi(x)$ 和它们的平方在区间 $[a, b]$ 上可积分. 证明哥西—布尼雅可夫斯基不等式

$$\left\{ \int_a^b \varphi(x) \psi(x) dx \right\}^2 \leq \int_a^b \varphi^2(x) dx \int_a^b \psi^2(x) dx.$$

证 证法一: 我们有

$$\begin{aligned} &\left(\int_a^b \varphi^2(x) dx \right) \cdot \left(\int_a^b \psi^2(x) dx \right) - \left(\int_a^b \varphi(x) \psi(x) dx \right)^2 \\ &= \frac{1}{2} \left(\int_a^b \varphi^2(x) dx \right) \cdot \left(\int_a^b \psi^2(y) dy \right) \\ &\quad + \frac{1}{2} \left(\int_a^b \psi^2(x) dx \right) \cdot \left(\int_a^b \varphi^2(y) dy \right) \\ &\quad - \left(\int_a^b \varphi(x) \psi(x) dx \right) \cdot \left(\int_a^b \varphi(y) \psi(y) dy \right) \end{aligned}$$

$$= \frac{1}{2} \int_a^b \left\{ \int_a^b [\varphi(x)\psi(y) - \psi(x)\varphi(y)]^2 dx \right\} dy \geq 0,$$

故

$$\left\{ \int_a^b \varphi(x)\psi(x) dx \right\}^2 \leq \int_a^b \varphi^2(x) dx \cdot \int_a^b \psi^2(x) dx.$$

证法二: 考虑积分

$$\int_a^b [\varphi(x) - \lambda\psi(x)]^2 dx,$$

其中 λ 为任意实数. 从而

$$\begin{aligned} \int_a^b \varphi^2(x) dx - 2\lambda \int_a^b \varphi(x)\psi(x) dx \\ + \lambda^2 \int_a^b \psi^2(x) dx \geq 0. \end{aligned}$$

这是关于变数 λ 的不等式, 左端是二次三项式, 于是其判别式

$$\begin{aligned} \left\{ \int_a^b \varphi(x)\psi(x) dx \right\}^2 - \int_a^b \varphi^2(x) dx \\ \int_a^b \psi^2(x) dx \leq 0, \end{aligned}$$

即

$$\begin{aligned} \left\{ \int_a^b \varphi(x)\psi(x) dx \right\}^2 \\ \leq \int_a^b \varphi^2(x) dx \cdot \int_a^b \psi^2(x) dx. \end{aligned}$$

2332. 设函数 $f(x)$ 在闭区间 $[a, b]$ 上连续可微分且 $f(a) = 0$, 证明不等式

$$M^2 \leq (b-a) \int_a^b f'^2(x) dx,$$

其中 $M = \sup_{a \leq x \leq b} |f(x)|$.

证 设 x 为 (a, b) 上任一点, 则利用哥西 — 布尼雅可夫斯基不等式得到

$$\left\{ \int_a^x f(x) dx \right\}^2 \leq \int_a^x 1 \cdot dx \cdot \int_a^x f'^2(x) dx,$$

即

$$\begin{aligned} f^2(x) &= [f(x) - f(a)]^2 \leq (x - a) \int_a^x f'^2(x) dx \\ &\leq (b - a) \int_a^b f'^2(x) dx. \end{aligned}$$

由此可知

$$M^2 = \sup_{x \in [a, b]} f^2(x) \leq (b - a) \int_a^b f'^2(x) dx.$$

2333. 证明等式:

$$\lim_{n \rightarrow \infty} \int_n^{n+p} \frac{\sin x}{x} dx = 0.$$

证 证法一: 应用第一中值定理, 知

$$\lim_{n \rightarrow \infty} \int_n^{n+p} \frac{\sin x}{x} dx = \lim_{n \rightarrow \infty} \frac{\sin \xi_n}{\xi_n} \cdot p = 0,$$

其中 ξ_n 为界于 n 与 $n + p$ 之间的某值.

证法二: 应用第二中值定理, 得

$$\begin{aligned} \left| \int_n^{n+p} \frac{\sin x}{x} dx \right| &= \frac{1}{n} \left| \int_n^{n+p} \sin x dx \right| \\ &= \frac{1}{n} |\cos n - \cos \xi'_n| \leq \frac{2}{n} \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

其中 ξ'_n 是界于 n 与 $n + p$ 之间的某值. 于是

$$\lim_{n \rightarrow \infty} \int_n^{n+p} \frac{\sin x}{x} dx = 0.$$

§ 4. 广 义 积 分

1° 函数的广义可积分性 若函数 $f(x)$ 于每一个有穷区间 $[a, b]$ 上依寻常的意义是可积分的, 则可定义

$$\int_a^{+\infty} f(x) dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) dx. \quad (1)$$

若函数 $f(x)$ 于点 b 的邻域内无界且于每一个区间 $(a, b - \varepsilon)$ ($\varepsilon > 0$) 内依寻常的意义是可积分的, 则取

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{b-\varepsilon} f(x) dx. \quad (2)$$

若极限(1)或(2)存在, 则对应的积分称为收敛的, 在相反的情形则称为发散的.

2° 哥西准则 积分(1)收敛的充要条件为对于任意的 $\varepsilon > 0$, 存在有数 $b = b(\varepsilon)$, 当 $b' > b$ 及 $b'' > b$ 时, 下面的不等式成立

$$\left| \int_{b'}^{b''} f(x) dx \right| < \varepsilon.$$

同样地对形状为(2)的积分可述出哥西准则.

3° 绝对收敛的判别法 若 $|f(x)|$ 是广义可积分的, 则函数 $f(x)$ 的对应的积分(1)或(2)称为绝对收敛的, 而且显然也是收敛的积分.

比较判别法 I. 设当 $x \geq a$ 时 $|f(x)| \leq F(x)$.

若 $\int_a^{+\infty} F(x) dx$ 收敛, 则积分 $\int_a^{+\infty} f(x) dx$ 绝对收敛.

比较判别法 II. 若 $\varphi(x) > 0$ 及当 $x \rightarrow +\infty$ 时,

$$\varphi(x) = O^*[\psi(x)],$$

则积分 $\int_a^{+\infty} \varphi(x) dx$ 及 $\int_a^{+\infty} \psi(x) dx$ 同时收敛或同时发散. 就特别情形来说, 若当 $x \rightarrow +\infty$ 时, $\varphi(x) \sim \psi(x)$, 则上面的结果也成立.

比较判别法 III. (a) 设当 $x \rightarrow +\infty$ 时,

$$f(x) = O\left(\frac{1}{x^p}\right).$$

在这种情况下,当 $p > 1$ 时,积分(1)收敛;当 $p \leq 1$ 时,积分(1)发散.

(6) 设当 $x \rightarrow b - 0$ 时,

$$f(x) = O\left[\frac{1}{(b-x)^p}\right].$$

在这种情况下,当 $p < 1$ 时,积分(2)收敛;当 $p \geq 1$ 时,积分(2)发散.

4° 收敛性的较精密的判别法 若(1)当 $x \rightarrow +\infty$ 时,函数 $\varphi(x)$ 单调地趋近于零;(2)函数 $f(x)$ 有有界的原函数

$$F(x) = \int_a^x f(\xi) d\xi,$$

则积分

$$\int_a^{+\infty} f(x)\varphi(x)dx$$

收敛,但一般地说,并非绝对收敛.

特殊情形,若 $p > 0$,则积分

$$\int_a^{+\infty} \frac{\cos x}{x^p} dx \text{ 及 } \int_a^{+\infty} \frac{\sin x}{x^p} dx \quad (a > 0)$$

收敛.

5° 在哥西意义上的主值 若函数 $f(x)$ 对任意的 $\epsilon > 0$ 积分

$$\int_a^{c-\epsilon} f(x)dx \text{ 及 } \int_{c+\epsilon}^b f(x)dx \quad (a < c < b)$$

存在,则在哥西意义上的主值($V \cdot P \cdot$) 为

$$\begin{aligned} V \cdot P \cdot \int_a^b f(x)dx \\ = \lim_{\epsilon \rightarrow +0} \left[\int_a^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^b f(x)dx \right]. \end{aligned}$$

$$\text{相仿地, } V \cdot P \cdot \int_{-\infty}^{+\infty} f(x)dx = \lim_{a \rightarrow +\infty} \int_{-a}^a f(x)dx.$$

计算下列积分:

$$2334. \int_a^{+\infty} \frac{dx}{x^2} \quad (a > 0).$$

解 由于

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi.$$

2338. $\int_2^{+\infty} \frac{dx}{x^2 + x - 2}.$

解 由于

$$\begin{aligned} \lim_{b \rightarrow +\infty} \int_2^b \frac{dx}{x^2 + x - 2} &= \lim_{b \rightarrow +\infty} \left(\frac{1}{3} \ln \frac{x-1}{x+2} \right) \Big|_2^b \\ &= \frac{1}{3} \lim_{b \rightarrow +\infty} \left(\ln \frac{b-1}{b+2} + 2 \ln 2 \right) = \frac{2}{3} \ln 2, \end{aligned}$$

所以

$$\int_2^{+\infty} \frac{dx}{x^2 + x - 2} = \frac{2}{3} \ln 2.$$

2339. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + x + 1)^2}.$

解 $\int \frac{dx}{(x^2 + x + 1)^2}$

$$= \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C.$$

由于

$$\begin{aligned} &\lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{(x^2 + x + 1)^2} + \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{(x^2 + x + 1)^2} \\ &= \lim_{a \rightarrow -\infty} \left\{ \left(\frac{1}{3} + \frac{4}{3\sqrt{3}} \arctan \frac{1}{\sqrt{3}} \right) \right. \\ &\quad \left. - \left[\frac{2a+1}{3(a^2+a+1)} + \frac{1}{3\sqrt{3}} \arctan \frac{2a+1}{\sqrt{3}} \right] \right\} \\ &\quad + \lim_{b \rightarrow +\infty} \left\{ \left[\frac{2b+1}{3(b^2+b+1)} + \frac{4}{3\sqrt{3}} \arctan \frac{2b+1}{\sqrt{3}} \right] \right. \\ &\quad \left. - \left(\frac{1}{3} + \frac{4}{3\sqrt{3}} \arctan \frac{1}{\sqrt{3}} \right) \right\} \end{aligned}$$

$$= \frac{4\pi}{3\sqrt{3}},$$

所以

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + x + 1)^2} = \frac{4\pi}{3\sqrt{3}}.$$

*) 利用 1921 题的递推公式.

2340. $\int_0^{+\infty} \frac{dx}{1+x^3}.$

解 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{1+x^3} \\ &= \lim_{b \rightarrow +\infty} \left[\frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2x-1}{\sqrt{3}} \right]^{**} \Big|_0^b \\ &= \frac{2\pi}{3\sqrt{3}}, \end{aligned}$$

所以

$$\int_0^{+\infty} \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}.$$

*) 利用 1881 题的结果.

2341. $\int_0^{+\infty} \frac{x^2+1}{x^4+1} dx.$

解 由于

$$\begin{aligned} & \lim_{\substack{b \rightarrow +\infty \\ \epsilon \rightarrow +0}} \int_{\epsilon}^b \frac{x^2+1}{x^4+1} dx \\ &= \lim_{\substack{b \rightarrow +\infty \\ \epsilon \rightarrow +0}} \left(\frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{x^2-1}{x\sqrt{2}} \right)^{**} \Big|_{\epsilon}^b = \frac{\pi}{\sqrt{2}}, \end{aligned}$$

所以

$$\int_0^{+\infty} \frac{x^2+1}{x^4+1} dx = \frac{\pi}{\sqrt{2}}.$$

*) 利用 1712 题的结果.

$$2342. \int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}.$$

解 先求 $\int \frac{dx}{(2-x)\sqrt{1-x}}$. 设 $\sqrt{1-x} = t$, 则

$$x = 1 - t^2, dx = -2t dt, 2 - x = 1 + t^2.$$

代入得

$$\begin{aligned} \int \frac{dx}{(2-x)\sqrt{1-x}} &= -2 \int \frac{dt}{1+t^2} \\ &= -2 \operatorname{arc} \operatorname{tg} t + C \\ &= -2 \operatorname{arc} \operatorname{tg} \sqrt{1-x} + C. \end{aligned}$$

由于

$$\begin{aligned} &\lim_{\epsilon \rightarrow +0} \int_0^{1-\epsilon} \frac{dx}{(2-x)\sqrt{1-x}} \\ &= \lim_{\epsilon \rightarrow +0} \left(-2 \operatorname{arc} \operatorname{tg} \sqrt{1-x} \Big|_0^{1-\epsilon} \right) \\ &= -2 \lim_{\epsilon \rightarrow +0} \left[\operatorname{arc} \operatorname{tg} \sqrt{1-(1-\epsilon)} - \frac{\pi}{4} \right] \\ &= \frac{\pi}{2}, \end{aligned}$$

所以

$$\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}} = \frac{\pi}{2}.$$

$$2343. \int_1^{+\infty} \frac{dx}{x\sqrt{1+x^5+x^{10}}}.$$

解 设 $\sqrt{1+x^5+x^{10}} = t - x^5$. 则当 $1 \leq x < +\infty$ 时, $1 + \sqrt{3} \leq t < +\infty$, 代入得

$$\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^5+x^{10}}}$$

$$\begin{aligned}
&= \frac{2}{5} \int_{1+\sqrt{3}}^{+\infty} \frac{dt}{t^2-1} = \frac{1}{5} \ln \left| \frac{t-1}{t+1} \right|_{1+\sqrt{3}}^{+\infty} \\
&= \frac{1}{5} \ln 1 - \frac{1}{5} \ln \frac{\sqrt{3}}{2+\sqrt{3}} = \frac{1}{5} \ln \left(1 + \frac{2}{\sqrt{3}} \right).
\end{aligned}$$

*) 牛顿—莱不尼兹公式对于广义积分也成立. 例如

$$\int_a^{+\infty} f(x)dx = F(+\infty) - F(a) = F(x) \Big|_a^{+\infty},$$

其中 $F(+\infty)$ 是一个符号, 代表 $\lim_{x \rightarrow +\infty} F(x)$ (假定此极限存在有限), 下同, 不再说明.

2344. $\int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx.$

解 我们有

$$\begin{aligned}
\int \frac{x \ln x}{(1+x^2)^2} dx &= -\frac{1}{2} \int \ln x d\left(\frac{1}{1+x^2}\right) \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{x(1+x^2)} \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx \\
&= -\frac{\ln x}{2(1+x^2)} + \frac{1}{4} \ln \frac{x^2}{1+x^2} + C.
\end{aligned}$$

由于

$$\begin{aligned}
&\lim_{\substack{\epsilon \rightarrow +0 \\ b \rightarrow +\infty}} \int_{\epsilon}^b \frac{x \ln x}{(1+x^2)^2} dx \\
&= \lim_{\substack{\epsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left[-\frac{\ln x}{2(1+x^2)} + \frac{1}{4} \ln \frac{x^2}{1+x^2} \right] \Big|_{\epsilon}^b \\
&= \lim_{\substack{\epsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left[-\frac{\ln b}{2(1+b^2)} + \frac{\ln \epsilon}{2(1+\epsilon^2)} + \frac{1}{4} \ln \frac{b^2}{b^2+1} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\ln\frac{\epsilon^2}{\epsilon^2+1}\Big) \\
& = \lim_{\epsilon \rightarrow +0} \left[-\frac{\epsilon^2}{2(\epsilon^2+1)}\ln\epsilon + \frac{1}{4}\ln(\epsilon^2+1) \right] \\
& = 0,
\end{aligned}$$

所以

$$\int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx = 0.$$

注 $\epsilon \rightarrow +0$ 与 $b \rightarrow +\infty$ 的极限过程是独立的, 因此可分别取极限.

2345. $\int_0^{+\infty} \frac{\arctg x}{(1+x^2)^{\frac{3}{2}}} dx.$

解 设 $x = \operatorname{tg} t$, 则

$$\begin{aligned}
\int_0^{+\infty} \frac{\arctg x}{(1+x^2)^{\frac{3}{2}}} dx &= \int_0^{\frac{\pi}{2}} \frac{t \sec^2 t dt}{\sec^3 t} \\
&= (t \sin t + \cos t) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1.
\end{aligned}$$

2346. $\int_0^{+\infty} e^{-ax} \cos bx dx \quad (a > 0).$

$$\begin{aligned}
\text{解 } \int_0^{+\infty} e^{-ax} \cos bx dx &= \left(\frac{-a \cos bx + b \sin bx}{a^2 + b^2} e^{-ax} \right)^{*)} \Big|_0^{+\infty} \\
&= \frac{a}{a^2 + b^2}.
\end{aligned}$$

*) 利用 1828 题的结果.

2347. $\int_0^{+\infty} e^{-ax} \sin bx dx \quad (a > 0).$

解 $\int_0^{+\infty} e^{-ax} \sin bx dx$

$$= \left(\frac{-a \sin bx - b \cos bx}{a^2 + b^2} e^{-ax} \right)^{**} \Big|_0^{+\infty}$$

$$= \frac{b}{a^2 + b^2}.$$

*) 利用 1829 题的结果.

利用递推公式计算下列广义积分(n 为自然数):

$$2348. \quad I_n = \int_0^{+\infty} x^n e^{-x} dx.$$

$$\begin{aligned} \text{解} \quad I_n &= \int_0^{+\infty} x^n d(-e^{-x}) \\ &= -x^n e^{-x} \Big|_0^{+\infty} + n \int_0^{+\infty} x^{n-1} e^{-x} dx \\ &= n \int_0^{+\infty} x^{n-1} e^{-x} dx = n I_{n-1}, \end{aligned}$$

即 $I_n = n I_{n-1}$. 利用此递推公式及

$$I_0 = \int_0^{+\infty} e^{-x} dx = 1$$

容易得到

$$I_n = n(n-1)\cdots 2 \cdot 1 I_0 = n!.$$

$$2349^+. \quad I_n = \int_{-\infty}^{+\infty} \frac{dx}{(ax^2 + 2bx + c)^n} (ac - b^2 > 0).$$

$$\begin{aligned} \text{解} \quad I_n &= \frac{ax+b}{2(n-1)(ac-b^2)(ax^2+2bx+c)^{n-1}} \Big|_{-\infty}^{+\infty} \\ &\quad + \frac{2n-3}{n-1} \cdot \frac{a}{2(ac-b^2)} I_{n-1}^{**} \\ &= \frac{2n-3}{2(n-1)} \cdot \frac{a}{ac-b^2} I_{n-1}, \end{aligned}$$

即

$$I_n = \frac{2n-3}{2(n-1)} \cdot \frac{a}{ac-b^2} I_{n-1} (n > 1).$$

$$\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty} \frac{dx}{ax^2 + 2bx + c} \\
&= \frac{\operatorname{sgn} a}{\sqrt{ac - b^2}} \operatorname{arc\,tg} \frac{|a| \left(x + \frac{b}{a} \right)}{\sqrt{ac - b^2}} \bigg|_{-\infty}^{+\infty} \\
&= \frac{\pi \operatorname{sgn} a}{\sqrt{ac - b^2}}.
\end{aligned}$$

利用递推公式及 I_1 容易得到

$$\begin{aligned}
I_n &= \frac{(2n-3)(2n-5)\cdots 3\cdot 1}{(2n-2)(2n-4)\cdots 4\cdot 2} \cdot \frac{\pi a^{n-1} \operatorname{sgn} a}{(ac - b^2)^{n-\frac{1}{2}}} \\
&= \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{\pi a^{n-1} \operatorname{sgn} a}{(ac - b^2)^{n-\frac{1}{2}}}
\end{aligned}$$

*) 利用 1921 题的结果.

$$2350^+. \quad I_n = \int_1^{+\infty} \frac{dx}{x(x+1)\cdots(x+n)}.$$

解 由于 $x^{n+1} \cdot \frac{1}{x(x+1)\cdots(x+n)} \rightarrow 1$ (当 $x \rightarrow +\infty$ 时), 且 $n+1 > 1$, 所以积分 I_n 收敛.

其次, 我们来计算 I_n . 由于

$$\begin{aligned}
&\frac{1}{x(x+1)\cdots(x+n)} \\
&= \frac{1}{n!x} - \frac{1}{(n-1)!(x+1)} \\
&\quad + \frac{1}{2!(n-2)!(x+2)} \\
&\quad - \cdots + (-1)^k \frac{1}{k!(n-k)!(x+k)} \\
&\quad + \cdots + (-1)^n \frac{1}{n!(x+n)},
\end{aligned}$$

所以

$$\begin{aligned}
 I_n &= \frac{1}{n!} \int_1^{+\infty} \sum_{k=0}^n C_n^k (-1)^k \frac{bx}{x+k} \\
 &= \frac{1}{n!} \sum_{k=0}^n (-1)^k C_n^k \ln(x+k) \Big|_1^{+\infty},
 \end{aligned}$$

其中 C_n^k 为从 n 个元素中每次取 k 个的组合数.

对于 \bar{n} , 不论是偶数还是奇数, 用上限代入 (此处理解为趋近于无穷时的极限) 后均为零. 事实上, 当 $n = 2m$ 时,

$$\begin{aligned}
 &\sum_{k=0}^{2m} (-1)^k C_{2m}^k \ln(x+k) \\
 &= \ln \frac{x \cdot (x+2)^{\binom{2}{2m}} \cdots (x+2m)^{\binom{2m}{2m}}}{(x+1)^{\binom{1}{2m}} (x+3)^{\binom{3}{2m}} \cdots (x+2m-1)^{\binom{2m-1}{2m}}}.
 \end{aligned}$$

由于

$$1 + C_{2m}^2 + \cdots + C_{2m}^{2m} = C_{2m}^1 + C_{2m}^3 + \cdots + C_{2m}^{2m-1},$$

所以, 当 $m \rightarrow +\infty$ 时

$$\sum_{k=0}^{2m} (-1)^k C_{2m}^k \ln(x+k) \longrightarrow \ln 1 = 0;$$

当 $n = 2m - 1$ 时,

$$\begin{aligned}
 &\sum_{k=0}^{2m-1} (-1)^k C_{2m-1}^k \ln(x+k) \\
 &= \ln \frac{x(x+2)^{\binom{2}{2m-1}} \cdots (x+2m-2)^{\binom{2m-2}{2m-1}}}{(x+1)^{\binom{1}{2m-1}} (x+3)^{\binom{3}{2m-1}} \cdots (x+2m-2)^{\binom{2m-1}{2m-1}}} \\
 &\longrightarrow 0 \text{ (当 } m \rightarrow +\infty \text{ 时)}.
 \end{aligned}$$

最后我们得到

$$I_n = \frac{1}{n!} \sum_{k=0}^n (-1)^{k+1} C_n^k \ln(1+k).$$

$$2351. I_n = \int_0^1 \frac{x^n dx}{\sqrt{(1-x)(1+x)}}.$$

解 由于 $\sqrt{1-x} \cdot \frac{x^n}{\sqrt{(1-x)(1+x)}} \rightarrow \frac{1}{2}$

(当 $x \rightarrow 1-0$ 时),

且 $p = \frac{1}{2} < 1$, 所以积分 I_n 收敛.

其次, 设 $x = \sin t$, 则

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n t \, dt \\ &= \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & \text{当 } n = 2k \text{ 时;}^*) \\ \frac{(2k-2)!!}{(2k-1)!!}, & \text{当 } n = 2k-1 \text{ 时.} \end{cases} \end{aligned}$$

*) 利用 2281 题的结果.

$$2352. I_n = \int_0^{+\infty} \frac{dx}{\operatorname{ch}^{n+1} x}.$$

解 设 $x = \ln \left(\operatorname{tg} \frac{t}{2} \right)$, 则

当 $0 \leq x < +\infty$ 时, $\frac{\pi}{2} \leq t \leq \pi$,

$$\begin{aligned} I_n &= \int_0^{+\infty} \frac{dx}{\operatorname{ch}^{n+1} x} = \int_{\frac{\pi}{2}}^{\pi} \sin^n t \, dt = \int_0^{\frac{\pi}{2}} \cos^n u \, du \\ &= \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & \text{当 } n = 2k \text{ 时;}^*) \\ \frac{(2k-2)!!}{(2k-1)!!}, & \text{当 } n = 2k-1 \text{ 时.} \end{cases} \end{aligned}$$

*) 利用 2282 题的结果.

$$2353. \quad (a) \int_0^{\frac{\pi}{2}} \ln \sin x \, dx; \quad (b) \int_0^{\frac{\pi}{2}} \ln \cos x \, dx.$$

于是, $2A = A - \frac{\pi}{2} \ln 2$, $A = -\frac{\pi}{2} \ln 2$, 即

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, dx = \int_0^{\frac{\pi}{2}} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2.$$

2354. 求:

$$\int_E e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx,$$

其中 E 表区间 $(0, +\infty)$ 中使被积分式有意义的一切 x 值所成之集合.

$$\begin{aligned} \text{解} \quad & \int_E e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx \\ &= \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx, * \end{aligned}$$

$$\text{对于广义积分} \int_{2k\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \cdot \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx$$

作如下处理:

$$\begin{aligned} & \int_{2k\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx \\ &= \int_{2k\pi}^{(2k+\frac{1}{4})\pi} e^{-\frac{x}{2}} \frac{\cos x - \sin x}{\sqrt{\sin x}} dx \\ & \quad + \int_{(2k+\frac{1}{4})\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{\sin x - \cos x}{\sqrt{\sin x}} dx \\ &= 2e^{-\frac{x}{2}} \sqrt{\sin x} \Big|_{2k\pi}^{(2k+\frac{1}{4})\pi} - 2e^{-\frac{x}{2}} \sqrt{\sin x} \Big|_{(2k+\frac{1}{4})\pi}^{(2k+1)\pi} \\ &= 2 \sqrt[4]{8} \cdot e^{-k\pi} \cdot e^{-\frac{\pi}{8}}. \end{aligned}$$

* 记号 $\sum_{k=0}^{\infty} S_k$ 理解为极限 $\lim_{n \rightarrow +\infty} \sum_{k=0}^n S_k$. 以后题解中不再说明.

由于

$$\sum_{k=0}^n 2 \sqrt[4]{8} \cdot e^{-\frac{\pi}{8}} e^{-k\pi} = 2 \sqrt[4]{8} \cdot e^{-\frac{\pi}{8}} \cdot \frac{1 - e^{-(n+1)\pi}}{1 - e^{-\pi}}.$$

当 $n \rightarrow +\infty$ 时, 上式的极限为 $2 \sqrt[4]{8} \cdot e^{-\frac{\pi}{8}} \cdot \frac{1}{1 - e^{-\pi}}$.

于是,

$$\int_E e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx = \frac{2 \sqrt[4]{8} e^{-\frac{\pi}{8}}}{1 - e^{-\pi}}.$$

2355. 证明等式

$$\int_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx = \frac{1}{a} \int_0^{+\infty} f(\sqrt{x^2 + 4ab}) dx,$$

其中 $a > 0, b > 0$ (假定等式左端的积分有意义).

证 设 $ax - \frac{b}{x} = t$, 则

当 $0 < x < +\infty$ 时, $-\infty < t < +\infty$,

$$ax + \frac{b}{x} = \sqrt{t^2 + 4ab}.$$

将此二式相加得

$$x = \frac{1}{2a} (t + \sqrt{t^2 + 4ab}).$$

从而有

$$dx = \frac{1}{2a} \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt.$$

代入欲证的等式左端, 得

$$\begin{aligned} & \int_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx \\ &= \frac{1}{2a} \int_{-\infty}^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2a} \int_{-\infty}^0 f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\
&\quad + \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\
&= \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{\sqrt{t^2 + 4ab} - t}{\sqrt{t^2 + 4ab}} dt \\
&\quad + \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\
&= \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \\
&\quad \cdot \frac{\sqrt{t^2 + 4ab} - t + \sqrt{t^2 + 4ab} + t}{\sqrt{t^2 + 4ab}} dt \\
&= \frac{1}{a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) dt,
\end{aligned}$$

于是

$$\int_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx = \frac{1}{a} \int_0^{+\infty} f(\sqrt{x^2 + 4ab}) dx.$$

2356. 数

$$M[f] = \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(\xi) d\xi$$

称为函数 $f(x)$ 在区间 $(0, +\infty)$ 上的平均值, 求下列函数的平均值:

$$(a) f(x) = \sin^2 x + \cos^2(x\sqrt{2});$$

$$(b) f(x) = \arctg x; \quad (B) f(x) = \sqrt{x} \sin x.$$

解 (a) 由于

$$\int_0^x [\sin^2 \xi + \cos^2(\xi\sqrt{2})] d\xi$$

$$\begin{aligned}
&= \int_0^x \left(\frac{1 - \cos 2\xi}{2} + \frac{1 + \cos(2\xi \sqrt{2})}{2} \right) d\xi \\
&= x - \frac{1}{4} \sin 2x + \frac{1}{4\sqrt{2}} \sin(2x \sqrt{2}),
\end{aligned}$$

所以

$$\begin{aligned}
M[f] &= \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x [\sin^2 \xi + \cos^2(\xi \sqrt{2})] d\xi \\
&= \lim_{x \rightarrow +\infty} \left[1 - \frac{1}{4x} \sin 2x \right. \\
&\quad \left. + \frac{1}{4x\sqrt{2}} \sin(2x \sqrt{2}) \right] \\
&= 1;
\end{aligned}$$

$$\begin{aligned}
(6) \quad M[f] &= \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x \arctan \xi d\xi \\
&= \lim_{x \rightarrow +\infty} \frac{1}{x} \left[x \arctan x - \frac{1}{2} \ln(1 + x^2) \right] \\
&= \frac{\pi}{2} - \lim_{x \rightarrow +\infty} \frac{\ln(1 + x^2)}{2x} \\
&= \frac{\pi}{2} - \lim_{x \rightarrow +\infty} \frac{2x}{2(1 + x^2)} = \frac{\pi}{2};
\end{aligned}$$

(B) 利用第二中值定理, 得

$$\begin{aligned}
\int_0^x \sqrt{\xi} \sin \xi d\xi &= \sqrt{x} \int_c^x \sin \xi d\xi \\
&= \sqrt{x} (\cos c - \cos x) \quad (0 \leq c \leq x),
\end{aligned}$$

于是,

$$\begin{aligned}
M[f] &= \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x \sqrt{\xi} \sin \xi d\xi \\
&= \lim_{x \rightarrow +\infty} \frac{\cos c - \cos x}{\sqrt{x}} = 0.
\end{aligned}$$

2357. 求:

$$(a) \lim_{x \rightarrow 0} x \int_x^1 \frac{\cos t}{t^2} dt; \quad (6) \lim_{x \rightarrow \infty} \frac{\int_0^x \sqrt{1+t^4} dt}{x^3};$$

$$(B) \lim_{x \rightarrow +0} \frac{\int_x^{+\infty} t^{-1} e^{-t} dt}{\ln \frac{1}{x}};$$

$$(r) \lim_{x \rightarrow +0} x^a \int_x^1 \frac{f(t)^{**})}{t^{a+1}} dt,$$

其中 $a > 0$, $f(t)$ 为闭区间 $[0, 1]$ 上的连续函数.

解 (a) 由于

$$1 - \frac{t^2}{2} \leq \cos t \leq 1,$$

所以

$$\int_x^1 \frac{1 - \frac{t^2}{2}}{t^2} dt \leq \int_x^1 \frac{\cos t}{t^2} dt \leq \int_x^1 \frac{dt}{t^2},$$

计算得

$$-\frac{3}{2} + \frac{x}{2} + \frac{1}{x} \leq \int_x^1 \frac{\cos t}{t^2} dt \leq -1 + \frac{1}{x}.$$

又由于

$$\lim_{x \rightarrow 0} x \left(-\frac{3}{2} + \frac{x}{2} + \frac{1}{x} \right) = 1$$

及

$$\lim_{x \rightarrow 0} x \left(-1 + \frac{1}{x} \right) = 1,$$

故最后得到

$$\lim_{x \rightarrow 0} x \int_x^1 \frac{\cos t}{t^2} dt = 1;$$

(6) 由于

$$t^2 < \sqrt{1+t^4},$$

所以

$$\int_0^x \sqrt{1+t^4} dt > \int_0^x t^2 dt = \frac{x^3}{3},$$

从而当 $x \rightarrow +\infty$ 时, $\int_0^x \sqrt{1+t^4} dt \rightarrow +\infty$.

利用洛比塔法则,得

$$\lim_{x \rightarrow +\infty} \frac{\int_0^x \sqrt{1+t^4} dt}{x^3} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1+x^4}}{3x^3} = \frac{1}{3};$$

(B) 由于 $\lim_{t \rightarrow +0} t \cdot (t^{-1}e^{-t}) = 1$, 故广义积分

$\int_0^{+\infty} t^{-1}e^{-t} dt$ 发散. 从而, 所求的极限是 $\frac{\infty}{\infty}$ 型未定式. 利

用洛比塔法则,得

$$\lim_{x \rightarrow +0} \frac{\int_x^{+\infty} t^{-1}e^{-t} dt}{\ln \frac{1}{x}} = \lim_{x \rightarrow +0} \frac{-e^{-x} \cdot x^{-1}}{-\frac{1}{x}} = 1;$$

(r) 由于 $f(t)$ 在 $t=0$ 处右连续, 故对于任意给定的 $\epsilon > 0$, 总存在 $\delta' > 0$, 使当 $0 < t < \delta'$ 时, 恒有

$$|f(t) - f(0)| < \frac{\alpha\epsilon}{2}.$$

今又取 $0 < \delta < \delta'$, 使当 $0 < x < \delta$ 时, 有

$$\left| x^\alpha \int_{\delta'}^1 \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right| < \frac{\epsilon}{2}.$$

于是, 当 $0 < x < \delta$ 时, 就有

$$\begin{aligned} & \left| x^\alpha \int_x^1 \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right| \\ &= \left| x^\alpha \int_x^{\delta'} \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right| \end{aligned}$$

$$\begin{aligned}
& + x^a \int_x^1 \frac{f(t) - f(0)}{t^{a+1}} dt \Big| \\
& \leq \frac{\alpha \epsilon}{2} \cdot x^a \int_x^{\delta'} \frac{dt}{t^{a+1}} + \frac{\epsilon}{2} \\
& = \frac{\epsilon}{2} x^a \left(\frac{1}{x^a} - \frac{1}{\delta'^a} \right) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

故

$$\lim_{x \rightarrow +0} x^a \int_x^1 \frac{f(t) - f(0)}{t^{a+1}} dt = 0,$$

最后得到

$$\begin{aligned}
\lim_{x \rightarrow +0} x^a \int_x^1 \frac{f(t)}{t^{a+1}} dt &= \lim_{x \rightarrow +0} x^a \int_x^1 \frac{f(0)}{t^{a+1}} dt \\
&= \lim_{x \rightarrow +0} x^a f(0) \left[-\frac{1}{a} t^{-a} \right] \Big|_x^1 \\
&= \lim_{x \rightarrow +0} x^a f(0) \left(\frac{1}{ax^a} - \frac{1}{a} \right) = \frac{f(0)}{a}.
\end{aligned}$$

*) 原题(B)(r) 中 $x \rightarrow +0$ 误印为 $x \rightarrow 0$.

研究下列积分的收敛性:

2358. $\int_0^{+\infty} \frac{x^2 dx}{x^4 - x^2 + 1}.$

解 由于 $x^2 \cdot \frac{x^2}{x^4 - x^2 + 1} \rightarrow 1$ (当 $x \rightarrow +\infty$ 时),

所以积分 $\int_0^{+\infty} \frac{x^2 dx}{x^4 - x^2 + 1}$ 收敛.

2359. $\int_1^{+\infty} \frac{dx}{x \sqrt[3]{x^2 + 1}}.$

解 由于 $x^{\frac{5}{3}} \cdot \frac{1}{x \sqrt[3]{x^2 + 1}} \rightarrow 1$ (当 $x \rightarrow +\infty$ 时)

所以积分 $\int_1^{+\infty} \frac{dx}{x \sqrt[3]{x^2 + 1}}$ 收敛.

$$2360. \int_0^2 \frac{dx}{\ln x}.$$

解 当 $0 < x < 1$ 时 $\ln x < 0$, 由于

$$\lim_{x \rightarrow 1-0} (1-x) \cdot \frac{1}{-\ln x} = \lim_{x \rightarrow 1-0} \frac{-1}{-\frac{1}{x}} = 1,$$

所以积分 $\int_0^1 \frac{dx}{\ln x}$ 发散, 从而积分 $\int_0^2 \frac{dx}{\ln x}$ 也发散.

$$2361. \int_0^{+\infty} x^{p-1} e^{-x} dx.$$

解 将积分分成 $\int_0^{+\infty} x^{p-1} e^{-x} dx = \int_0^1 x^{p-1} e^{-x} dx$
 $+ \int_1^{+\infty} x^{p-1} e^{-x} dx.$

对于积分 $\int_0^1 x^{p-1} e^{-x} dx$, 由于

$$x^{1-p} \cdot (x^{p-1} e^{-x}) \rightarrow 1 \text{ (当 } x \rightarrow +0 \text{ 时)}, \text{ 故当 } p > 0$$

时 (从而 $1-p < 1$), 积分 $\int_0^1 x^{p-1} e^{-x} dx$ 收敛.

对于积分 $\int_1^{+\infty} x^{p-1} e^{-x} dx$, 由于

$$x^2 \cdot (x^{p-1} e^{-x}) = \frac{x^{p+1}}{e^x} \rightarrow 0 \text{ (当 } x \rightarrow +\infty \text{ 时)},$$

故对于一切 p 值, 积分 $\int_1^{+\infty} x^{p-1} e^{-x} dx$ 恒收敛.

于是, 当 $p > 0$ 时, 积分

$$\int_0^{+\infty} x^{p-1} e^{-x} dx$$

收敛.

$$2362. \int_0^1 x^p \ln^q \frac{1}{x} dx.$$

解 将积分分成 $\int_0^1 x^p \ln^q \frac{1}{x} dx = \int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$
 $+ \int_{\frac{1}{2}}^1 x^p \ln^q \frac{1}{x} dx.$

对于积分 $\int_{\frac{1}{2}}^1 x^p \ln^q \frac{1}{x} dx$, 由于

$$\begin{aligned} \lim_{x \rightarrow 1-0} (1-x)^{-q} \cdot x^p \ln^q \frac{1}{x} &= \lim_{x \rightarrow 1-0} x^p \left(\frac{\ln \frac{1}{x}}{1-x} \right)^q \\ &= \left[\lim_{x \rightarrow 1-0} \frac{\ln \frac{1}{x}}{1-x} \right]^q \\ &= \left[\lim_{x \rightarrow 1-0} \frac{x \left(-\frac{1}{x^2} \right)}{-1} \right]^q = 1, \end{aligned}$$

故 $\int_{\frac{1}{2}}^1 x^p \ln^q \frac{1}{x} dx$ 当 $-q < 1$ (即 $q > -1$) 时收敛, 当
 $-q \geq 1$ (即 $q \leq -1$) 时发散. 于是, 当 $q \leq -1$ 时,
 $\int_0^1 x^p \ln^q \frac{1}{x} dx$ 必发散. 故下面可在 $q > -1$ 的假定下来

讨论 $\int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$.

若 $p > -1$, 可取 $\tau > 0$ 充分小, 使 $p - \tau > -1$.
 于是

$$\lim_{x \rightarrow +0} x^{-p+\tau} \cdot x^p \ln^q \frac{1}{x} = \lim_{x \rightarrow +0} \frac{\left(\ln \frac{1}{x} \right)^q}{\left(\frac{1}{x} \right)^\tau} = 0.$$

由于 $-p + \tau < 1$, 故此时 $\int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$ 收敛;

$$\int_0^{+\infty} \frac{x^n}{1+x^n} dx (n \geq 0)$$

收敛.

2364. $\int_0^{+\infty} \frac{\arctg ax}{x^n} dx (a \neq 0).$

解 由于 $\arctg ax = -\arctg(-ax)$, 故可设 $a > 0$,

先考虑积分 $\int_0^1 \frac{\arctg ax}{x^n} dx$. 由于

$$\begin{aligned} \lim_{x \rightarrow +0} x^{n-1} \cdot \frac{\arctg ax}{x^n} &= \lim_{x \rightarrow +0} \frac{\arctg ax}{x} \\ &= \lim_{x \rightarrow +0} \frac{a}{1+a^2x^2} = a, \end{aligned}$$

故积分 $\int_0^1 \frac{\arctg ax}{x^n} dx$ 仅当 $n-1 < 1$ 即当 $n < 2$ 时收敛.

再考虑积分 $\int_1^{+\infty} \frac{\arctg ax}{x^n} dx$. 由于

$$x^n \cdot \frac{\arctg ax}{x^n} \rightarrow \frac{\pi}{2} \text{ (当 } x \rightarrow +\infty \text{ 时)},$$

故积分 $\int_1^{+\infty} \frac{\arctg ax}{x^n} dx$ 仅当 $n > 1$ 时收敛.

于是, 仅当 $1 < n < 2$ 时, 积分

$$\int_1^{+\infty} \frac{\arctg ax}{x^n} dx (a \neq 0)$$

收敛.

2365. $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx.$

解 先考虑积分 $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$. 当 $n > 1$ 时, 取 $a > 0$ 充分小, 使 $n-a > 1$. 由于

$$x^{n-1} \cdot \frac{\ln(1+x)}{x^n} = \frac{\ln(1+x)}{x} \rightarrow 0$$

(当 $x \rightarrow +\infty$ 时),

故此时积分 $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$ 收敛. 当 $n \leq 1$ 时, 由于

$$x^n \cdot \frac{\ln(1+x)}{x^n} \rightarrow +\infty \text{ (当 } x \rightarrow +\infty \text{ 时),}$$

故此时积分 $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$ 发散.

再考虑积分 $\int_0^1 \frac{\ln(1+x)}{x^n} dx$. 由于

$$\lim_{x \rightarrow +0} x^{n-1} \cdot \frac{\ln(1+x)}{x^n} = \lim_{x \rightarrow +0} \frac{\ln(1+x)}{x} = 1,$$

故积分 $\int_0^1 \frac{\ln(1+x)}{x^n} dx$ 仅当 $n-1 < 1$ 即当 $n < 2$ 时收敛.

于是, 仅当 $1 < n < 2$ 时, 积分

$$\int_0^{+\infty} \frac{\ln(1+x)}{x^n} dx$$

收敛.

2366. $\int_0^{+\infty} \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx. (n \geq 0).$

解 先考虑积分 $\int_0^1 \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$. 由于

$$\lim_{x \rightarrow +0} x^{-m-1} \cdot \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} = \frac{1}{2} \lim_{x \rightarrow +0} \frac{\operatorname{arc} \operatorname{tg} x}{x}$$

$$= \frac{1}{2} \lim_{x \rightarrow +0} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{2},$$

故积分 $\int_0^1 \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$ 仅当 $-m-1 < 1$ 即当 $m >$

-2 时收敛.

再考虑积分 $\int_1^{+\infty} \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$. 由于

$$\begin{aligned} & x^{n-m} \cdot \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} \\ &= \frac{x^n \operatorname{arc} \operatorname{tg} x}{2+x^n} \rightarrow \frac{\pi}{2} \quad (\text{当 } x \rightarrow +\infty \text{ 时}), \end{aligned}$$

故积分 $\int_1^{+\infty} \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$ 仅当 $n-m > 1$ 时收敛.

于是, 仅当 $m > -2$ 且 $n-m > 1$ 时, 积分

$$\int_0^{+\infty} \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} (n \geq 0)$$

收敛.

2367. $\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx \quad (n \geq 0).$

解 当 $a \neq 0$ 时, 设 $f(x) = \cos ax, g(x) = \frac{1}{1+x^n}$,

则对于任意的 $A > 0$, 均有 $\left| \int_0^A f(x) dx \right| \leq \frac{2}{a}$; 其次, 当 $n > 0$ 时, $g(x)$ 单调下降且趋于零 ($n \rightarrow +\infty$). 从而得知积分

$$\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx$$

收敛. 至于当 $n = 0$ 时, 积分显然发散.

当 $a = 0$ 时, 由于

$$x^n \cdot \frac{1}{1+x^n} \rightarrow 1 \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故积分 $\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx$ 仅当 $n > 1$ 时收敛.

于是, 当 $a \neq 0, n > 0$ 及 $a = 0, n > 1$ 时, 积分

$$\int_0^{+\infty} \frac{\cos ax}{1+x^a} dx.$$

收敛.

2368. $\int_0^{+\infty} \frac{\sin x}{x} dx.$

解 方法一:

$$\frac{\sin^2 x}{x} = \frac{1 - \cos 2x}{2x} = \frac{1}{2} \left(\frac{1}{x} - \frac{\cos 2x}{x} \right).$$

积分 $\int_1^{+\infty} \frac{dx}{x}$ 显然发散.

又因对于任意的 $A > 1$, $\left| \int_1^A \cos 2x dx \right| \leq 2$, 且当 x

$\rightarrow +\infty$ 时, $\frac{1}{x}$ 单调地趋于零, 故积分

$$\int_1^{+\infty} \frac{\cos 2x}{x} dx \text{ 收敛.}$$

于是, 积分 $\int_1^{+\infty} \frac{\sin^2 x}{x} dx$ 发散, 从而积分

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx$$

发散.

方法二:

$$\begin{aligned} \int_0^{+\infty} \frac{\sin^2 x}{x} dx &= \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin^2 x}{x} dx \\ &= \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin^2 t}{t + n\pi} dt \geq \frac{1}{\pi} \int_0^{\pi} \sin^2 t dt \cdot \sum_{n=0}^{\infty} \frac{1}{n+1} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}. \end{aligned}$$

由于不论 N 取多大, 只要取 $p = N$, 就有

$$\begin{aligned}
\sum_{k=N+1}^{N+p} \frac{1}{k} &= \sum_{i=N+1}^{2N} \frac{1}{k} \\
&= \frac{1}{N+1} + \cdots + \frac{1}{2N} > \underbrace{\frac{1}{2N} + \frac{1}{2N} + \cdots + \frac{1}{2N}}_{N \uparrow} \\
&= \frac{1}{2N} \cdot N = \frac{1}{2},
\end{aligned}$$

故递增数列

$$S_n = \sum_{k=1}^n \frac{1}{k} \quad (n = 1, 2, \cdots)$$

的极限 $\lim_{n \rightarrow \infty} S_n$ 是 $+\infty$, 即 $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$.

于是, 积分

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx$$

发散.

2369. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}.$

解 先考虑积分 $\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^p x \cos^q x}$, 对于任何 q 值, 由于

$$\begin{aligned}
&\lim_{x \rightarrow +0} x^p \cdot \frac{1}{\sin^p x \cos^q x} \\
&= \lim_{x \rightarrow +0} \left(\frac{x}{\sin x} \right)^p \cdot \lim_{x \rightarrow +0} \left(\frac{1}{\cos^q x} \right) = 1,
\end{aligned}$$

故积分 $\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^p x \cos^q x}$ 仅当 $p < 1$ (q 为任意值) 时收敛.

再考虑积分 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$, 对于任何 p 值, 由于

$$\begin{aligned}
& \lim_{x \rightarrow \frac{\pi}{2}-0} \left(\frac{\pi}{2} - x \right)^q \cdot \frac{1}{\sin^p x \cos^q x} \\
&= \lim_{x \rightarrow \frac{\pi}{2}-0} \left(\frac{\frac{\pi}{2} - x}{\cos x} \right)^q \cdot \lim_{x \rightarrow \frac{\pi}{2}-0} \left(\frac{1}{\sin^p x} \right) \\
&= \lim_{t \rightarrow +0} \left(\frac{t}{\sin t} \right)^q = 1,
\end{aligned}$$

故积分 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$ 仅当 $q < 1$ (p 为任意值) 时收敛.

于是, 当 $p < 1$ 且 $q < 1$ 时, 积分

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$$

收敛.

2370. $\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}.$

解 先考虑积分 $\int_0^{\frac{1}{2}} \frac{x^n dx}{\sqrt{1-x^2}}$. 由于

$$\lim_{x \rightarrow +0} \left(x^{-n} \cdot \frac{x^n}{\sqrt{1-x^2}} \right) = 1,$$

故积分 $\int_0^{\frac{1}{2}} \frac{x^n dx}{\sqrt{1-x^2}}$ 仅当 $-n < 1$ 即当 $n > -1$ 时收敛.

再考虑积分 $\int_{\frac{1}{2}}^1 \frac{x^n}{\sqrt{1-x^2}} dx$. 对于任意的 n ,

由于

$$\lim_{x \rightarrow 1-0} \left(\sqrt{1-x} \cdot \frac{x^n}{\sqrt{1-x^2}} \right)$$

$$= \lim_{x \rightarrow 1-0} \frac{x^n}{\sqrt{1+x}} = \frac{1}{\sqrt{2}},$$

故积分 $\int_{\frac{1}{2}}^1 \frac{x^n}{\sqrt{1-x^2}} dx$ 恒收敛.

于是, 当 $n > -1$ 时, 积分

$$\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}$$

收敛.

2371. $\int_0^{+\infty} \frac{dx}{x^p + x^q}.$

解 先考虑积分 $\int_0^1 \frac{dx}{x^p + x^q}$. 不妨设 $\min(p, q) = p$, 由于

$$\lim_{x \rightarrow +0} \left(x^p \cdot \frac{1}{x^p + x^q} \right) = \lim_{x \rightarrow +0} \frac{1}{1 + x^{q-p}} = 1,$$

故积分 $\int_0^1 \frac{dx}{x^p + x^q}$ 仅当 $p < 1$, 即当 $\min(p, q) < 1$ 时收敛.

再考虑积分 $\int_1^{+\infty} \frac{dx}{x^p + x^q}$. 不妨设 $\max(p, q) = q$,

由于

$$\lim_{x \rightarrow +\infty} \left(x^q \cdot \frac{1}{x^p + x^q} \right) = \lim_{x \rightarrow +\infty} \frac{1}{x^{-(q-p)} + 1} = 1,$$

故积分 $\int_1^{+\infty} \frac{dx}{x^p + x^q}$ 仅当 $q > 1$ 即当 $\max(p, q) > 1$ 时收敛.

于是, 当 $\min(p, q) < 1$ 且 $\max(p, q) > 1$ 时, 积分

$$\int_0^{+\infty} \frac{dx}{x^p + x^q}$$

收斂.

$$2372. \int_0^1 \frac{\ln x}{1-x^2} dx.$$

解 先考虑积分 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x^2} dx$. 由于

$$\lim_{x \rightarrow +0} \left(\sqrt{x} \cdot \frac{\ln x}{1-x^2} \right) = 0,$$

故积分 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x^2} dx$ 收斂.

再考虑积分 $\int_{\frac{1}{2}}^1 \frac{\ln x}{1-x^2} dx$. 由于

$$\lim_{x \rightarrow 1-0} \left(\sqrt{1-x} \cdot \frac{\ln x}{1-x^2} \right) = 0,$$

故积分 $\int_{\frac{1}{2}}^1 \frac{\ln x}{1-x^2} dx$ 收斂.

于是, 积分

$$\int_0^1 \frac{\ln x}{1-x^2} dx$$

收斂.

$$2373. \int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\sqrt{x}} dx.$$

解 由于

$$\begin{aligned} & \lim_{x \rightarrow +0} \left(x^{\frac{5}{6}} \cdot \frac{\ln(\sin x)}{\sqrt{x}} \right) \\ &= \lim_{x \rightarrow +0} \left(\left(\frac{x}{\sin x} \right)^{\frac{1}{3}} \cdot \sqrt[3]{\sin x} \ln(\sin x) \right) = 0, \end{aligned}$$

故积分 $\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\sqrt{x}} dx$ 收斂.

$$2374. \int_1^{+\infty} \frac{dx}{x^p \ln^q x}.$$

解 先考虑 $\int_1^2 \frac{dx}{x^p \ln^q x}$. 对于任意的 p , 由于

$$\begin{aligned} & \lim_{x \rightarrow 1+0} \left[(x-1)^q \cdot \frac{1}{x^p \ln^q x} \right] \\ &= \lim_{x \rightarrow 1+0} \left[\frac{1}{x^p} \cdot \left(\frac{x-1}{\ln x} \right)^q \right] = \left(\lim_{x \rightarrow 1+0} \frac{x-1}{\ln x} \right)^q \\ &= \left[\lim_{x \rightarrow 1+0} \frac{1}{x} \right]^q = 1, \end{aligned}$$

故积分 $\int_1^2 \frac{dx}{x^p \ln^q x}$ 仅当 $q < 1$ 且 p 为任意值时收敛.

再考虑积分 $\int_2^{+\infty} \frac{dx}{x^p \ln^q x}$. 如果 $p > 1$, 取 $\alpha > 0$ 充分小, 使 $p - \alpha > 1$, 则对于任意的 q , 由于

$$\lim_{x \rightarrow +\infty} \left(x^{p-\alpha} \cdot \frac{1}{x^p \ln^q x} \right) = \lim_{x \rightarrow +\infty} \left(\frac{1}{x^\alpha \ln^q x} \right) = 0,$$

故积分 $\int_2^{+\infty} \frac{dx}{x^p \ln^q x}$ 收敛; 如果 $p \leq 1, q < 1$, 由于

$$\begin{aligned} & \int_2^{+\infty} \frac{dx}{x^p \ln^q x} \geq \int_2^{+\infty} \frac{dx}{x \ln^q x} \\ &= \frac{(\ln x)^{1-q}}{1-q} \Big|_2^{+\infty} = +\infty, \end{aligned}$$

故积分 $\int_2^{+\infty} \frac{dx}{x^p \ln^q x}$ 发散.

于是, 当 $p > 1$ 且 $q < 1$ 时, 积分

$$\int_1^{+\infty} \frac{dx}{x^p \ln^q x}$$

收敛.

$$2375. \int_e^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}.$$

(3) 当 $p < 1$ 时, 取 $\delta > 0$ 充分小, 使 $p + \delta < 1$.
 对于任意的 q 和 r , 由于

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{x^{p+\delta}}{x^p (\ln x)^q (\ln \ln x)^r} \\ &= \lim_{x \rightarrow +\infty} \frac{x^\delta}{(\ln x)^q (\ln \ln x)^r} = +\infty, \end{aligned}$$

故此时积分 $\int_3^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$ 发散.

于是, 当 $p > 1, q$ 是任意的, $r < 1$ 和当 $p = 1, q > 1, r < 1$ 时, 积分

$$\int_e^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$$

收敛.

$$2376. \int_{-\infty}^{+\infty} \frac{dx}{|x-a_1|^{p_1} |x-a_2|^{p_2} \cdots |x-a_n|^{p_n}}.$$

解 首先, 被积函数关于 $\frac{1}{x}$ 是 $\sum_{i=1}^n P_i$ 级无穷小 (当 $x \rightarrow \pm \infty$ 时).

其次 (不妨设当 $i \neq j$ 时, $a_i \neq a_j$),

$$\begin{aligned} & \lim_{x \rightarrow a_i} \left(|x-a_i|^p \cdot \frac{1}{|x-a_1|^{p_1} |x-a_2|^{p_2} \cdots |x-a_n|^{p_n}} \right) \\ &= c_i, 0 < c_i < +\infty (i = 1, 2, \cdots, n), \end{aligned}$$

故积分 $\int_{-\infty}^{+\infty} \frac{dx}{|x-a_1|^{p_1} |x-a_2|^{p_2} \cdots |x-a_n|^{p_n}}$ 仅当

$\sum_{i=1}^n p_i > 1$ 且 $p_i < 1 (i = 1, 2, \cdots, n)$ 时收敛.

$$2377. \int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx,$$

式中 $P_m(x)$ 及 $P_n(x)$ 为次数分别为 m 及 n 的互质的多

项式.

解 当 $P_n(x) = 0$ 在 $[0, +\infty)$ 上有根 λ 并设其重数为 $r(\geq 1)$ 时, 由于 $P_n(x)$ 与 $P_m(x)$ 互质, 故 λ 不是 $P_m(x)$ 的根. 从而有

$$\lim_{x \rightarrow \lambda} \left[(x - \lambda)^r \cdot \frac{P_m(x)}{P_n(x)} \right] = a \neq 0,$$

而且显然在点 λ 的右(左)近旁, $\frac{P_m(x)}{P_n(x)}$ 都保持定号. 由于 $r \geq 1$, 故积分发散. 由于

$$\lim_{x \rightarrow +\infty} \left[x^{n-m} \cdot \frac{P_m(x)}{P_n(x)} \right] = b \neq 0,$$

故积分 $\int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx$ 仅当 $n - m > 1$ 即当 $n > m + 1$ 时收敛.

于是, 当 $P_n(x)$ 在区间 $[0, +\infty)$ 内无根且 $n > m + 1$ 时, 积分

$$\int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx$$

收敛.

研究下列积分的绝对收敛性和条件收敛性:

2378. $\int_1^{+\infty} \frac{\sin x}{x} dx.$

解 对于任意的 $A > 1$, 由于 $\left| \int_1^A \sin x dx \right| \leq 2$, 且

当 $x \rightarrow +\infty$ 时, $\frac{1}{x}$ 单调地趋于零, 故积分

$$\int_1^{+\infty} \frac{\sin x}{x} dx$$

收敛. 而积分 $\int_0^1 \frac{\sin x}{x} dx$ 是普通的定积分
 $\left(\frac{\sin x}{x} \text{ 在 } x=0 \text{ 有可去间断点, 故补充定义其值为 } 1 \right.$
 $\left. \text{后, } \frac{\sin x}{x} \text{ 可视为 } [0, 1] \text{ 上的连续函数} \right)$, 故积分

$$\int_0^{+\infty} \frac{\sin x}{x} dx$$

收敛. 但它不是绝对收敛的. 事实上, 当 $x > 0$ 时,
 $\left| \frac{\sin x}{x} \right| \geq \frac{\sin^2 x}{x}$, 由 2368 题知, 积分 $\int_0^{+\infty} \frac{\sin^2 x}{x} dx$ 发散,
 故积分 $\int_0^{+\infty} \left| \frac{\sin x}{x} \right| dx$ 发散.

2379. $\int_0^{+\infty} \frac{\sqrt{x} \cos x}{x+100} dx.$

解 对于任意的 $A > 0$, 由于 $\left| \int_0^A \cos x dx \right| \leq 2$, 且

当 $x \rightarrow +\infty$ 时, $\frac{\sqrt{x}}{x+100}$ 单调地趋于零, 故积分

$$\int_0^{+\infty} \frac{\sqrt{x} \cos x}{x+100} dx$$

收敛. 但它不是绝对收敛的. 事实上, 由于

$$\begin{aligned} \frac{\sqrt{x} |\cos x|}{x+100} &\geq \frac{\sqrt{x} \cos^2 x}{x+100} \\ &= \frac{1}{2} \left(\frac{\sqrt{x}}{x+100} - \frac{\sqrt{x} \cos 2x}{x+100} \right), \end{aligned}$$

且 $\lim_{x \rightarrow +\infty} \left(x^{\frac{1}{2}} \cdot \frac{\sqrt{x}}{x+100} \right) = 1$, 故积分 $\int_0^{+\infty} \frac{\sqrt{x}}{x+100} dx$

发散. 仿照前半段证明, 可知 $\int_0^{+\infty} \frac{\sqrt{x} \cos 2x}{x+100} dx$

收敛. 从而, 积分 $\int_0^{+\infty} \frac{\sqrt{x} \cos^2 x}{x+100} dx$ 发散. 于是, 积分

$$\int_0^{+\infty} \frac{\sqrt{x} |\cos x|}{x+100} dx$$

发散.

2380. $\int_0^{+\infty} x^p \sin(x^q) dx (q \neq 0).$

解 设 $t = x^q$, 则 $dx = \frac{1}{q} t^{\frac{1}{q}-1} dt$. 于是

$$\int_0^{+\infty} x^p \sin(x^q) dx = \frac{1}{|q|} \int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt.$$

先考虑积分 $\int_0^1 t^{\frac{p+1}{q}-1} \sin t dt$. 由于

$$\lim_{t \rightarrow +0} \left(t^{-\frac{p+1}{q}} \cdot t^{\frac{p+1}{q}-1} \sin t \right) = \lim_{t \rightarrow +0} \frac{\sin t}{t} = 1,$$

故积分 $\int_0^1 t^{\frac{p+1}{q}-1} \sin t dt$ 仅当 $-\frac{p+1}{q} < 1$, 即当

$\frac{p+1}{q} > -1$ 时收敛, 又由于被积函数在 $[0, 1]$ 上非负, 故也是绝对收敛的.

再考虑积分 $\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$. 如果 $\frac{p+1}{q} < 1$,

则由于对任意的 $A > 1$, $\left| \int_1^A \sin t dt \right| \leq 2$ 且 $t^{\frac{p+1}{q}-1}$

单调地趋于零 (当 $t \rightarrow +\infty$ 时), 故此时积分

$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$ 收敛. 如果 $\frac{p+1}{q} = 1$, 则积分

$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$ 显然发散, 从而积分 $\int_0^{+\infty} t^{\frac{p+1}{q}-1}$

• $\sin t dt$ 也发散. 如果 $\frac{p+1}{q} > 1$, 则由于 $\lim_{t \rightarrow +\infty} t^{\frac{p+1}{q}-1} = +\infty$, 故对任给的 $A > 0$, 总存在自然数 N , 使有 $2N\pi + \frac{\pi}{4} > A$, 且当 $t > 2N\pi + \frac{\pi}{4}$ 时, $t^{\frac{p+1}{q}-1} > \sqrt{2}$. 今取

$$A' = 2N\pi + \frac{\pi}{4}, A'' = 2N\pi + \frac{\pi}{2},$$

则有

$$\left| \int_{A'}^{A''} t^{\frac{p+1}{q}-1} \sin t dt \right| > \sqrt{2} \left| \int_{A'}^{A''} \sin t dt \right| = 1,$$

它不可能小于任给的 $\epsilon (0 < \epsilon < 1)$, 因而, 积分

$$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt,$$

发散, 从而积分

$$\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$$

也发散.

于是, 仅当 $-1 < \frac{p+1}{q} < 1$ 时, 积分

$$\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$$

收敛, 且当 $\frac{p+1}{q} > -1$ 时, 积分

$$\int_0^1 t^{\frac{p+1}{q}-1} \sin t dt$$

绝对收敛.

下面我们考虑积分 $\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$ 的绝对收

敛性,分三种情形讨论:

(1) 当 $\frac{p+1}{q} < 0$ 时,由于

$$|t^{\frac{p+1}{q}-1} \sin t| \leq t^{\frac{p+1}{q}-1} (1 \leq t < +\infty),$$

且 $\int_1^{+\infty} t^{\frac{p+1}{q}-1} dt$ 收敛,故当 $\frac{p+1}{q} < 0$ 时,积分

$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$ 绝对收敛;

(2) 当 $\frac{p+1}{q} = 0$ 时,由于

$$\begin{aligned} & \int_1^{+\infty} |t^{\frac{p+1}{q}-1} \sin t| dt \\ &= \int_1^{+\infty} \left| \frac{\sin t}{t} \right| dt = +\infty \end{aligned}$$

故此时积分不绝对收敛(但条件收敛);

(3) 当 $\frac{p+1}{q} > 0$ 时,由于

$$\begin{aligned} & \int_1^{+\infty} |t^{\frac{p+1}{q}-1} \sin t| dt \\ & \geq \int_1^{+\infty} \frac{|\sin t|}{t} dt = +\infty, \end{aligned}$$

故此时积分也不是绝对收敛的.

于是,当 $-1 < \frac{p+1}{q} < 0$ 时,积分

$$\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$$

绝对收敛.

最后我们得到:当 $-1 < \frac{p+1}{q} < 0$ 时,积分

$$\int_0^{+\infty} x^p \sin(x^q) dx$$

绝对收敛;当 $0 \leq \frac{p+1}{q} < 1$ 时,积分条件收敛.

$$2381. \int_0^{+\infty} \frac{x^p \sin x}{1+x^q} dx \quad (q \geq 0).$$

解 先考虑积分 $\int_0^1 \frac{x^p \sin x}{1+x^q} dx$. 由于

$$\begin{aligned} & \lim_{x \rightarrow +0} \left(x^{-1-p} \cdot \frac{x^p \sin x}{1+x^q} \right) \\ &= \lim_{x \rightarrow +0} \left(\frac{\sin x}{x} \cdot \frac{1}{1+x^q} \right) = 1, \end{aligned}$$

故积分 $\int_0^1 \frac{x^p \sin x}{1+x^q} dx$ 仅当 $-1-p < 1$ 即当 $p > -2$ 时收敛,且是绝对收敛的.

再考虑积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$. (1) 若 $p \geq q$, 则对任何 $A > 1$, 必存在正整数 N , 使 $2N\pi + \frac{\pi}{4} > A$ 且当 $x \geq 2N\pi + \frac{\pi}{4}$ 时, 恒有 $\frac{x^p}{1+x^q} > \frac{1}{3}$. 于是, 对 $A' = 2N\pi + \frac{\pi}{4}$, $A'' = 2N\pi + \frac{\pi}{2}$, 有

$$\begin{aligned} & \left| \int_{A'}^{A''} \frac{x^p}{1+x^q} \sin x dx \right| > \frac{1}{3} \int_{A'}^{A''} \sin x dx \\ &= \frac{\sqrt{2}}{6}, \end{aligned}$$

它不可能小于任给的 ϵ , 故积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 发散.

(2) 若 $p < q-1$, 取 $\alpha > 0$ 使 $p+\alpha < q-1$, 即 $q-p-\alpha > 1$, 由于

$$\lim_{x \rightarrow +\infty} x^{q-p-\alpha} \cdot \frac{x^p}{1+x^q} |\sin x|$$

$$\lim_{x \rightarrow +\infty} \frac{x^q}{1+x^q} \cdot \frac{|\sin x|}{x^q} = 0,$$

故积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 绝对收敛. (3) 现设 $q-1 \leq p < q$. 先证 $\int_1^{+\infty} \frac{x^p |\sin x|}{1+x^q} dx$ 发散. 事实上, 此时, 可取 $A_0 > 1$, 使当 $x \geq A_0$ 时, $\frac{x^{p+1}}{1+x^q} > \frac{1}{3}$;

$$\begin{aligned} \text{故 } \int_{A_0}^{+\infty} \frac{x^p |\sin x|}{1+x^q} dx &= \int_{A_0}^{+\infty} \frac{x^{p+1}}{1+x^q} \cdot \left| \frac{\sin x}{x} \right| dx \\ &\geq \frac{1}{3} \int_{A_0}^{+\infty} \left| \frac{\sin x}{x} \right| dx = +\infty, \end{aligned}$$

从而 $\int_1^{+\infty} \frac{x^p |\sin x|}{1+x^q} dx$ 发散.

再证 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 收敛. 事实上, 若 $q=0$, 则 $-1 \leq p < 0$, 此时积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$
 $= \frac{1}{2} \int_1^{+\infty} x^p \sin x dx$ 显然收敛; 若 $q > 0$, 由于
 $\left(\frac{x^p}{1+x^q} \right)' = \frac{x^{p-1} [p - (q-p)x^q]}{(1+x^q)^2} < 0$ (当 x 充分大时),

故当 $x \rightarrow +\infty$ 时, $\frac{x^p}{1+x^q}$ 单调递减趋于零. 而

$$\left| \int_1^A \sin x dx \right| = |\cos 1 - \cos A| \leq 2 \text{ 有界, 故积分}$$

$\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 收敛. 总之, 我们证明了: 当 $q-1 \leq p < q$ 时, $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 条件收敛.

于是,最后得结论:积分 $\int_0^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 当 $p > -2, q > p+1$ 时绝对收敛;当 $p > -2, p < q \leq p+1$ 时条件收敛.

$$2382. \int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx.$$

解 当 $n \leq 0$ 时,积分显然是发散的.

当 $n > 0$ 时,首先考虑积分 $\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx (a > 1)$. 由于

$$\begin{aligned} & \int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx \\ &= \int_0^{+\infty} \frac{\left(1 - \frac{1}{x^2}\right) \sin\left(x + \frac{1}{x}\right)}{x^n \left(1 - \frac{1}{x^2}\right)} dx, \end{aligned}$$

而

$$\begin{aligned} & \left| \int_a^A \left(1 - \frac{1}{x^2}\right) \sin\left(x + \frac{1}{x}\right) dx \right| \\ & \left| \cos\left(a + \frac{1}{a}\right) - \cos\left(A + \frac{1}{A}\right) \right| \leq 2. \end{aligned}$$

又当 x 充分大时,有

$$\frac{d}{dx} x^n \left(1 - \frac{1}{x^2}\right) = nx^{n-3} \left(x^2 - \frac{n-2}{n}\right) > 0, \text{故当 } x$$

$$\begin{aligned} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} &\geq \frac{\sin^2\left(x + \frac{1}{x}\right)}{x^n} \\ &= \frac{1 - \cos\left(2x + \frac{2}{x}\right)}{2x^n}, \end{aligned}$$

而当 $0 < n \leq 1$ 时, 积分 $\int_a^{+\infty} \frac{dx}{x^n}$ 显然发散, 积分 $\int_0^{+\infty} \frac{\cos\left(2x + \frac{2}{x}\right)}{x^n} dx$ 收敛 (仿前半段证明), 故当 $0 < n \leq$

1 时, 积分 $\int_0^{+\infty} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$ 发散, 从而当 $0 < n \leq 1$ 时, 积分

$$\int_0^{+\infty} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$$

发散. 对于 $1 < n < 2$ 的情况, 可考虑对积分作变换 $x = \frac{1}{t}$, 则得

$$\begin{aligned} \int_0^a \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx \\ = \int_{\frac{1}{a}}^{+\infty} \frac{\left| \sin\left(t + \frac{1}{t}\right) \right|}{t^{2-n}} dt. \end{aligned}$$

仿前可知, 当 $0 < 2 - n \leq 1$ 即当 $1 \leq n < 2$ 时, 积分

$\int_0^a \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$ 发散. 从而, 当 $1 < n < 2$ 时, 积分

$$\int_0^{+\infty} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$$

发散

最后我们得到: 当 $0 < n < 2$ 时, 积分

$$\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$$

条件收敛.

$$2383^+. \int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx,$$

式中 $P_m(x)$ 及 $P_n(x)$ 为整多项式, 且若 $x \geq a$, $P_n(x) > 0$.

解 今仿 2381 题解之. 设

$$P_m(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m,$$

$$P_n(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n,$$

其中 m, n 是非负整数, $a_0 \neq 0, b_0 \neq 0$.

(1) 若 $n > m + 1$, 可取 $a > 0$ 充分小, 使 $n - a > m + 1$. 由于

$$\begin{aligned} & \lim_{x \rightarrow +\infty} x^{n-m-a} \cdot \left| \frac{P_m(x)}{P_n(x)} \sin x \right| \\ &= \lim_{x \rightarrow +\infty} \left| \frac{x^n P_m(x)}{x^m P_n(x)} \right| \cdot \frac{|\sin x|}{x^a} = 0, \end{aligned}$$

而 $n - m - a > 1$, 故积分 $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 绝对收敛.

(2) 若 $n = m + 1$. 我们证明此时 $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 条

件收敛. 事实上, 由于 $\lim_{x \rightarrow +\infty} \frac{xP_m(x)}{P_n(x)} = \frac{a_0}{b_0}$, 故存在 A_0

$> a$, 使当 $x \geq A_0$ 时, 恒有 $\left| \frac{xP_m(x)}{P_n(x)} \right| > \frac{|a_0|}{2|b_0|}$,

于是

$$\begin{aligned} & \int_{A_0}^{+\infty} \left| \frac{P_m(x)}{P_n(x)} \sin x \right| dx \\ &= \int_{A_0}^{+\infty} \left| \frac{xP_m(x)}{P_n(x)} \right| \cdot \left| \frac{\sin x}{x} \right| dx \\ &\geq \frac{|a_0|}{2|b_0|} \int_{A_0}^{+\infty} \left| \frac{\sin x}{x} \right| dx = +\infty, \end{aligned}$$

故 $\int_a^{+\infty} \left| \frac{P_m(x)}{P_n(x)} \sin x \right| dx$ 发散. 此外, 易知 ($n = m + 1$ 时)

$$\begin{aligned} \left(\frac{P_m(x)}{P_n(x)} \right)' &= \frac{1}{[P_n(x)]^2} \{ -a_0 b_0 x^{2m} \\ &\quad - 2a_1 b_0 x^{2m-1} + \cdots + (a_{m-1} b_{m+1} \\ &\quad - a_m b_m) \}, \end{aligned}$$

故若 $a_0 b_0 > 0$, 则当 x 充分大时, $\left(\frac{P_m(x)}{P_n(x)} \right)' < 0$, 函数

$\frac{P_m(x)}{P_n(x)}$ 减小; 若 $a_0 b_0 < 0$, 则当 x 充分大时, $\left(\frac{P_m(x)}{P_n(x)} \right)'$

> 0 , 函数 $\frac{P_m(x)}{P_n(x)}$ 增加. 总之, 当 $x \rightarrow +\infty$ 时, $\frac{P_m(x)}{P_n(x)}$

单调地趋于零. 又显然可知

$\left| \int_a^A \sin x dx \right| \leq 2$, 故积分 $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 收敛.

(3) 若 $n < m + 1$. 由于 n, m 都是非负整数, 故 $n \leq m$.

因此

$$\lim_{x \rightarrow +\infty} \frac{P_m(x)}{P_n(x)} = \begin{cases} \frac{a_0}{b_0}, & \text{若 } n = m; \\ +\infty, & \text{若 } n < m \text{ 且 } a_0 b_0 > 0; \\ -\infty, & \text{若 } n < m \text{ 且 } a_0 b_0 < 0. \end{cases}$$

于是, 存在 $A^* > a$ 及 $\tau > 0$, 使当 $x \geq A^*$ 时 $\frac{P_m(x)}{P_n(x)}$ 保

持定号且 $\left| \frac{P_m(x)}{P_n(x)} \right| > \tau$. 今对任何 $A > a$, 可取正整数

N , 使 $2N\pi + \frac{\pi}{4} \geq \max\{A, A^*\}$. 令 $A' = 2N\pi + \frac{\pi}{4}$,

$A'' = 2N\pi + \frac{\pi}{2}$, 则

$$\begin{aligned} \left| \int_{A'}^{A''} \frac{P_m(x)}{P_n(x)} \sin x dx \right| &> \tau \int_{A'}^{A''} \sin x dx \\ &= \frac{\tau \sqrt{2}}{2}, \end{aligned}$$

它不能小于任意的 $\varepsilon (0 < \varepsilon < \frac{\tau \sqrt{2}}{2})$, 故

$\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 发散.

最后, 我们得出: $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 当 $n > m + 1$

时绝对收敛; 当 $n = m + 1$ 时条件收敛.

2384. 若 $\int_a^{+\infty} f(x) dx$ 收敛, 则当 $x \rightarrow +\infty$ 时是否必有 $f(x) \rightarrow 0$? 研究例子:

$$(a) \int_0^{+\infty} \sin(x^2) dx; \quad (b) \int_0^{+\infty} (-1)^{[x^2]} dx.$$

解 不一定. 例如

(a) 积分 $\int_0^{+\infty} \sin(x^2) dx$ 收敛. 事实上, 它是 2380 题之特例: $p = 0, q = 2$. 但是, $\lim_{x \rightarrow +\infty} \sin(x^2)$ 不存在;

(6) 先证积分 $\int_0^{+\infty} (-1)^{[x^2]} dx$ 收敛. 事实上, 对任何 $A > 0$, 存在唯一的非负整数 n , 使 $\sqrt{n} \leq A < \sqrt{n+1}$. 显然 $A \rightarrow +\infty$ 相当于 $n \rightarrow \infty$. 当 $\sqrt{k} \leq x < \sqrt{k+1}$ (k —非负整数) 时, $[x^2] = k$. 于是

$$\begin{aligned} & \int_0^A (-1)^{[x^2]} dx \\ &= \sum_{k=0}^{n-1} \int_{\sqrt{k}}^{\sqrt{k+1}} (-1)^k dx + (-1)^n (A - \sqrt{n}) \\ &= \sum_{k=0}^{n-1} (-1)^k \frac{1}{\sqrt{k+1} + \sqrt{k}} + (-1)^n (A - \sqrt{n}). \end{aligned}$$

由于 $\frac{1}{\sqrt{k+1} + \sqrt{k}}$ 递减趋于 0 (当 $k \rightarrow \infty$ 时), 故

$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (-1)^k \frac{1}{\sqrt{k+1} + \sqrt{k}}$ 存在有限 (参看 2656 题前面的变号级数的莱布尼兹判别法), 设为 S . 又显然

$$\begin{aligned} & |(-1)^n (A - \sqrt{n})| < \sqrt{n+1} - \sqrt{n} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0 \text{ (当 } n \rightarrow \infty \text{ 时)}, \end{aligned}$$

故 $\lim_{A \rightarrow +\infty} \int_0^A (-1)^{[x^2]} dx = S$, 因此

积分 $\int_0^{+\infty} (-1)^{[x^2]} dx$ 收敛.

但显然 $\lim_{x \rightarrow +\infty} (-1)^{[x^2]}$ 不存在.

2385. 于 $[a, b]$ 上有定义的, 无界函数 $f(x)$ 可否把函数 $f(x)$ 的收敛广义积分

$$\int_a^b f(x) dx$$

看作对应的积分和式

$$\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

的极限? 式中 $x_i \leq \xi_i \leq x_{i+1}$ 及 $\Delta x_i = x_{i+1} - x_i$.

解 不能. 因为若 $c (a \leq c \leq b)$ 是瑕点, 则对于 $[a, b]$ 的任何分法, 不论其 $\max |\Delta x_i|$ 多么小, 当分法确定以

后, 设 $c \in [x_i, x_{i+1}]$, 则总可以取 ξ_i , 使 $\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$ 大

于任何预先给定的值. 因此, 当 $\max |\Delta x_i| \rightarrow 0$ 时,

$\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$ 不可能具有有限极限.

2386. 设:

$$\int_a^{+\infty} f(x) dx \quad (1)$$

收敛, 函数 $\varphi(x)$ 有界, 则积分

$$\int_a^{+\infty} f(x) \varphi(x) dx \quad (2)$$

是否必定收敛? 举出适当的例子.

若积分(1)绝对收敛, 问积分(2)的收敛性如何?

解 不. 例如, 积分

$$\int_0^{+\infty} \frac{\sin x}{x} dx$$

收敛^{*)}. 且 $\varphi(x) = \sin x$ 有界, 但是积分

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx$$

是发散的^{**))}.

若积分(1) 绝对收敛, $\varphi(x)$ 有界, 则积分(2) 一定是绝对收敛的. 事实上, 设 $|\varphi(x)| \leq L$, 则由不等式

$$|f(x)\varphi(x)| \leq L \cdot |f(x)|$$

及 $\int_a^{+\infty} |f(x)| dx$ 的收敛性即可获证.

*) 利用 2378 题的结果.

* *) 利用 2368 题的结果.

2387. 证明, 若 $\int_a^{+\infty} f(x) dx$ 收敛, $f(x)$ 为单调函数, 则

$$f(x) = o\left(\frac{1}{x}\right)^{**}).$$

证 不妨设 $f(x)$ 单调减小. 先证当 $x \geq a$ 时, $f(x) \geq 0$. 若不然, 则存在点 $c \geq a$, 使 $f(c) < 0$. 由于 $f(x)$ 单调减小, 故当 $x \geq c$ 时, $f(x) \leq f(c)$, 从而

$$\int_c^{+\infty} f(x) dx \leq \int_c^{+\infty} f(c) dx = -\infty.$$

因此, 积分

$$\int_a^{+\infty} f(x) dx$$

发散, 这与积分 $\int_a^{+\infty} f(x) dx$ 收敛矛盾. 于是, $f(x)$ 为非负的单调函数.

下面证明 $f(x) = o\left(\frac{1}{x}\right)$. 由于积分

$$\int_a^{+\infty} f(x) dx$$

收敛,故对任给的 $\varepsilon > 0$,总存在 $A > a$,使当 $x > A$ 时,恒有

$$\left| \int_{\frac{x}{2}}^x f(t) dt \right| < \frac{\varepsilon}{2}.$$

但是

$$\begin{aligned} \left| \int_{\frac{x}{2}}^x f(t) dt \right| &= \int_{\frac{x}{2}}^x f(t) dt \geqslant f(x) \cdot \left(x - \frac{x}{2} \right) \\ &= \frac{x}{2} f(x), \end{aligned}$$

故当 $x > A$ 时,

$$0 \leqslant x f(x) < \varepsilon,$$

即

$$\lim_{x \rightarrow +\infty} x f(x) = 0 \quad \text{或} \quad f(x) = o\left(\frac{1}{x}\right).$$

如果 $f(x)$ 单调增大,则可考虑 $-f(x)$ (它是单调减小的),同法可证得 $f(x) = o\left(\frac{1}{x}\right)$.

*) 原题为 $f(x) = O\left(\frac{1}{x}\right)$, 现在的结果更好.

2388. 设函数 $f(x)$ 于区间 $0 < x \leqslant 1$ 内是单调的函数,且在点 $x = 0$ 的邻域内是无界的,证明若

$$\int_0^1 f(x) dx$$

存在,则

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

证 设函数 $f(x)$ 在 $(0, 1]$ 上是单调下降的. 这时 $\lim_{x \rightarrow +0} f(x) = +\infty$. 先设 $f(x) \geq 0$ ($0 < x \leq 1$ 时).

由于积分

$$\int_0^1 f(x) dx$$

存在, 故把区间 $[0, 1]$ n 等分后, 即得

$$\begin{aligned} \int_0^1 f(x) dx &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \\ &< \int_0^{\frac{1}{n}} f(x) dx + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} \\ &< \int_0^{\frac{1}{n}} f(x) dx + \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n}. \end{aligned}$$

另一方面, 又有

$$\int_0^1 f(x) dx > \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n}.$$

从而就有

$$0 < \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) < \int_0^{\frac{1}{n}} f(x) dx.$$

由于 $\lim_{n \rightarrow \infty} \int_0^{\frac{1}{n}} f(x) dx = 0$, 故

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

如果不满足 $f(x) \geq 0$, 即 $f(x)$ 可正可负. 则函数 $\varphi(x) = f(x) - f(1)$ 满足 $\varphi(x) \geq 0$ ($0 < x \leq 1$), 且同样是单调下降, $\lim_{x \rightarrow +0} \varphi(x) = +\infty$. 故根据已证的结果, 知

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi\left(\frac{k}{n}\right) = \int_0^1 \varphi(x) dx,$$

$$\lim_{x \rightarrow +0} \int_{\frac{x}{2}}^x t^p f(t) dt = 0, \text{ 从而 } \lim_{x \rightarrow +0} x^{p+1} f(x) = 0.$$

再设不存在上述 δ . 于是, 根据 $f(x)$ 的递减性, 知当 $0 < x < a$ 时恒有 $f(x) < 0$. 于是, 当 $0 < x < \frac{a}{2}$ 时, 有

$$\begin{aligned} \int_x^{2x} t^p f(t) dt &\leq f(x) \int_x^{2x} t^p dt \\ &= C_p^* x^{p+1} f(x) < 0, \end{aligned}$$

$$\text{其中 } C_p^* = \begin{cases} \frac{2^{p+1} - 1}{p+1}, & \text{当 } p \neq -1 \text{ 时;} \\ \ln 2, & \text{当 } p = -1 \text{ 时.} \end{cases}$$

故 C_p^* 也是正的常数.

于是, $|x^{p+1} f(x)| < \frac{1}{C_p^*} \left| \int_x^{2x} t^p f(t) dt \right|$. 根据

$\int_0^a x^p f(x) dx$ 的存在性, 知

$$\lim_{x \rightarrow +0} \int_x^{2x} t^p f(t) dt = 0,$$

从而 $\lim_{x \rightarrow +0} x^{p+1} f(x) = 0$. 证完.

2390. 证明

$$(a) \quad V. P. \int_{-1}^1 \frac{dx}{x} = 0;$$

$$(6) \quad V. P. \int_0^{+\infty} \frac{dx}{1-x^2} = 0;$$

$$(B) \quad V. P. \int_{-\infty}^{+\infty} \sin x dx = 0.$$

证(a) 由于

$$\lim_{\epsilon \rightarrow +0} \left\{ \int_{-1}^{0-\epsilon} \frac{dx}{x} + \int_{0+\epsilon}^1 \frac{dx}{x} \right\}$$

$$= \lim_{\epsilon \rightarrow +0} (\ln \epsilon - \ln 1 + \ln 1 - \ln \epsilon) = 0,$$

所以

$$V. P. \int_{-1}^1 \frac{dx}{x} = 0;$$

(6) 由于

$$\begin{aligned} & \lim_{\substack{\epsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left(\int_0^{1-\epsilon} \frac{dx}{1-x^2} + \int_{1+\epsilon}^b \frac{dx}{1-x^2} \right) \\ &= \lim_{\substack{\epsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left(\frac{1}{2} \ln \left| \frac{2-\epsilon}{\epsilon} \right| + \frac{1}{2} \ln \left| \frac{1+b}{1-b} \right| \right. \\ & \quad \left. - \frac{1}{2} \ln \left| \frac{2+\epsilon}{\epsilon} \right| \right) \\ &= \frac{1}{2} \lim_{\epsilon \rightarrow +0} \ln \left| \frac{2-\epsilon}{2+\epsilon} \right| = 0 \end{aligned}$$

所以

$$V. P. \int_0^{+\infty} \frac{dx}{1-x^2} = 0;$$

(B) 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_{-b}^b \sin x dx = \lim_{b \rightarrow +\infty} (-\cos b + \cos b) \\ &= 0, \end{aligned}$$

所以

$$V. P. \int_{-\infty}^{+\infty} \sin x dx = 0.$$

2391. 证明: 当 $x \geq 0$ 且 $x \neq 1$ 时

$$\operatorname{li} x = V. P. \int_0^x \frac{d\xi}{\ln \xi}$$

存在*).

证 当 $0 \leq x < 1$ 时, 由于 $\lim_{\xi \rightarrow +0} \frac{1}{\ln \xi} = 0$, 故将 $\frac{1}{\ln x}$ 在 x

$= 0$ 处补充定义后成为连续函数, 于是积分存在.

当 $x > 1$ 时, 首先注意到下面这样一个结论: 当 $a < c < b$ 时,

$$\begin{aligned} V. P. \int_a^b \frac{dx}{x-c} \\ &= \lim_{\epsilon \rightarrow +0} \left(\int_a^{c-\epsilon} \frac{dx}{x-c} + \int_{c+\epsilon}^b \frac{dx}{x-c} \right) \\ &= \ln \frac{b-c}{c-a}. \end{aligned}$$

其次, 利用具比亚诺型余项的台劳公式, 有

$$\ln x = (x-1) + [\alpha(x)-1] \frac{(x-1)^2}{2},$$

式中 $\lim_{x \rightarrow 1} \alpha(x) = 0$. 由此即得

$$\frac{1}{\ln x} = \frac{1}{x-1} - \frac{\frac{1}{2}[\alpha(x)-1]}{1 + \frac{[\alpha(x)-1]}{2}(x-1)},$$

上述等式右端的第二项在 $x=1$ 的附近保持有界, 且对于任意的 x 值连续, 因而是可积分的. 第一项的“主值”如前所述, 它是存在的.

于是, 当 $x \geq 0$ 且 $x \neq 1$ 时, $\operatorname{li} x$ 存在.

*) 原题误为“当 $x \geq 0$ 时, ...”.

求下列积分:

2392. $V. P. \int_0^{+\infty} \frac{dx}{x^2-3x+2}.$

解 由于

$$\lim_{\substack{\epsilon \rightarrow +0 \\ \eta \rightarrow +0 \\ b \rightarrow +\infty}} \left(\int_0^{1-\epsilon} \frac{dx}{x^2-3x+2} + \int_{1+\epsilon}^{2-\eta} \frac{dx}{x^2-3x+2} \right)$$

$$\begin{aligned}
& + \int_{2+\eta}^b \frac{dx}{x^2 - 3x + 2} \Big) \\
& = \lim_{\substack{\epsilon \rightarrow +0 \\ \eta \rightarrow +0 \\ b \rightarrow +\infty}} \left(\ln \frac{\epsilon + 1}{\epsilon} - \ln 2 + \ln \frac{\eta}{1 - \eta} - \ln \frac{1 - \epsilon}{\epsilon} \right. \\
& \quad \left. + \ln \left| \frac{b - 2}{b - 1} \right| - \ln \frac{\eta}{1 + \eta} \right) \\
& = \lim_{\substack{\epsilon \rightarrow +0 \\ \eta \rightarrow +0}} \left(\ln \frac{\epsilon + 1}{1 - \epsilon} - \ln 2 + \ln \frac{1 + \eta}{1 - \eta} \right) \\
& = -\ln 2 = \ln \frac{1}{2},
\end{aligned}$$

所以

$$V. P. \int_0^{+\infty} \frac{dx}{x^2 - 3x + 2} = \ln \frac{1}{2}.$$

2393. $V. P. \int_{\frac{1}{2}}^2 \frac{dx}{x \ln x}.$

解 由于

$$\begin{aligned}
& \lim_{\epsilon \rightarrow +0} \left(\int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{x \ln x} + \int_{1+\epsilon}^2 \frac{dx}{x \ln x} \right) \\
& = \lim_{\epsilon \rightarrow +0} [\ln |\ln(1 - \epsilon)| - \ln(\ln 2) + \ln(\ln 2) - \ln |\ln(1 + \epsilon)|] \\
& = \lim_{\epsilon \rightarrow +0} \ln \left| \frac{\ln(1 - \epsilon)}{\ln(1 + \epsilon)} \right| = \ln \left| \lim_{\epsilon \rightarrow +0} \frac{\ln(1 - \epsilon)}{\ln(1 + \epsilon)} \right| \\
& = \ln \left| \lim_{\epsilon \rightarrow +0} \frac{\frac{-1}{1 - \epsilon}}{\frac{1}{1 + \epsilon}} \right| = \ln 1 = 0,
\end{aligned}$$

所以

$$V. P. \int_{\frac{1}{2}}^2 \frac{dx}{x \ln x} = 0.$$

$$2394. \quad V. P. \int_{-\infty}^{+\infty} \frac{1+x}{1+x^2} dx.$$

解 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_{-b}^b \frac{1+x}{1+x^2} dx \\ &= \lim_{b \rightarrow +\infty} \left[\arctan b - \arctan(-b) + \frac{1}{2} \ln(1+b^2) \right. \\ & \quad \left. - \frac{1}{2} \ln(1+b^2) \right] \\ &= 2 \lim_{b \rightarrow +\infty} \arctan b = \pi, \end{aligned}$$

所以

$$V. P. \int_{-\infty}^{+\infty} \frac{1+x}{1+x^2} dx = \pi.$$

$$2395. \quad V. P. \int_{-\infty}^{+\infty} \arctan x dx.$$

解 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_{-b}^b \arctan x dx \\ &= \lim_{b \rightarrow +\infty} \left[b \arctan b - (-b) \arctan(-b) \right. \\ & \quad \left. - \frac{1}{2} \ln(1+b^2) + \frac{1}{2} \ln(1+b^2) \right] = 0, \end{aligned}$$

所以

$$V. P. \int_{-\infty}^{+\infty} \arctan x dx = 0.$$

§ 5. 面积的计算法

1° 直角坐标系中的面积 由两条连续的曲线 $y = y_1(x)$ 和 $y = y_2(x)$ [$y_2(x) \geq y_1(x)$] 与 Ox 轴的两条垂线 $x = a$ 和 $x = b$ 所围成的面

积 $S = A_1A_2B_2B_1$ (图 4.14) 等于

$$S = \int_a^b (y_2(x) - y_1(x)) dx.$$

2° 参数形状表出的曲线所围成的面积 若 $x = x(t), y = y(t) (0 \leq t \leq T)$ 为一逐段平滑的简单封闭曲线 C 的参数方程式, 面积 S 表由此曲线所围在它左侧的面积 (图 4.15), 则

$$\begin{aligned} S &= - \int_0^T (y(t)x'(t) dt \\ &= \int_a^T x(t)y'(t) dt \end{aligned}$$

或

$$S = \frac{1}{2} \int_0^T (x(t)y'(t) - x'(t)y(t)) dt.$$

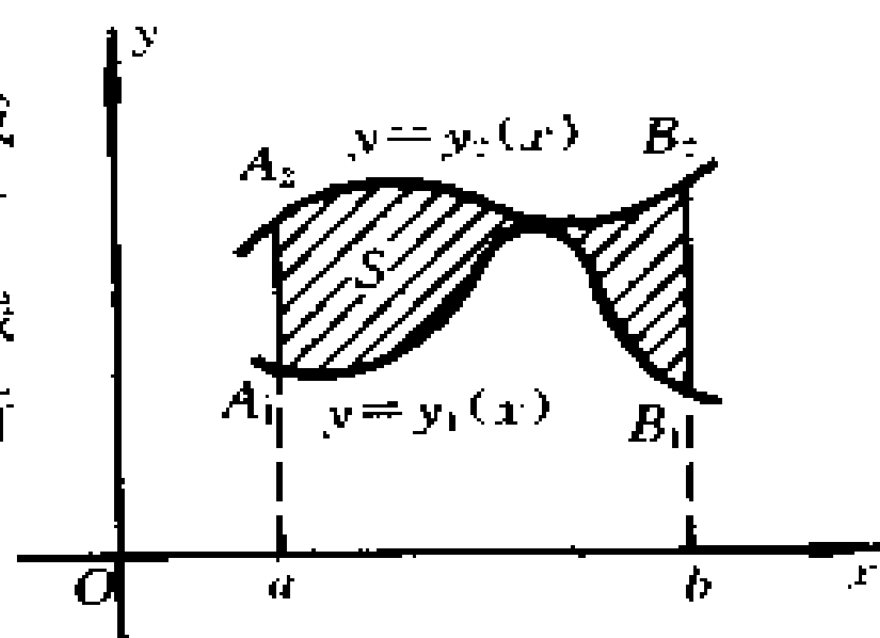


图 4.14

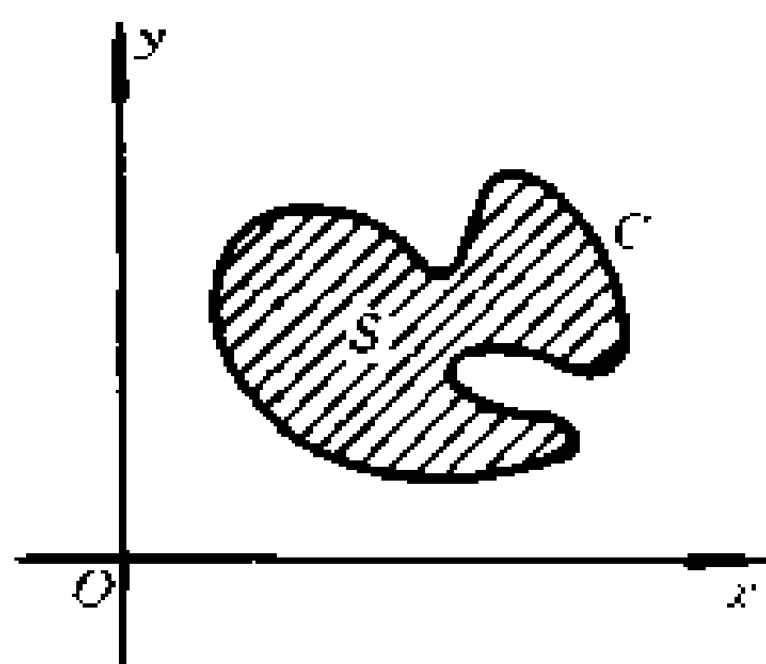


图 4.15

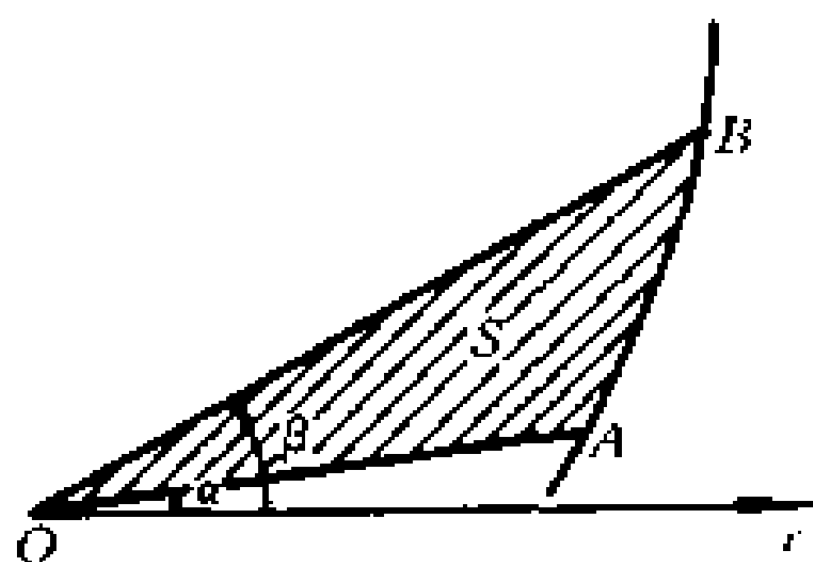


图 4.16

3° 极坐标系中的面积 由连续的曲线 $r = r(\varphi)$ 和两条半射线 $\varphi = \alpha$ 和 $\varphi = \beta$ 所围成的面积 $S = OAB$ (图 4.16) 等于

$$S = \frac{1}{2} \int_{\alpha}^{\beta} r^2(\varphi) d\varphi.$$

2396. 证明 正抛物线拱的面积等于

$$S = \frac{2}{3}bh,$$

式中 b 为底, h 为拱的高(图 4.17).

证 设抛物线的方程为

$$y = Ax^2 + Bx + C,$$

则当 $x = \pm \frac{b}{2}$ 时, 得

$$y = \frac{Ab^2}{4} \pm \frac{Bb}{2} + C = 0;$$

当 $x = 0$ 时, 得

$$y = C = h.$$

解之得

$$A = -\frac{4h}{b^2}, B = 0.$$

从而

$$y = -\frac{4h}{b^2}x^2 + h.$$

于是, 所求的面积为

$$\begin{aligned} S &= 2 \int_0^{\frac{b}{2}} \left(h - \frac{4h}{b^2}x^2 \right) dx \\ &= 2 \left(hx - \frac{4h}{3b^2}x^3 \right) \Big|_0^{\frac{b}{2}} = \frac{2}{3}bh. \end{aligned}$$

求下列直角坐标方程所表曲线围成的面积*).

2397. $ax = y^2, ay = x^2.$

解 如图 4.18 所示, 交点为

$$A(a, a) \text{ 及 } O(0, 0).$$

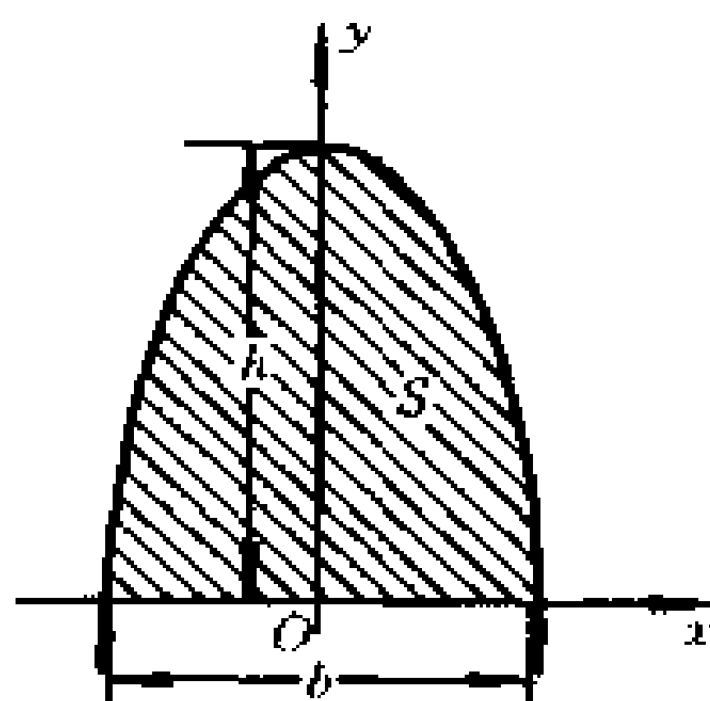


图 4.17

*) 在第四章的这一节和以后各节都把一切的参数当作是正的.

所求的面积为

$$\begin{aligned} S &= \int_0^a \left(\sqrt{ax} - \frac{x^2}{a} \right) dx \\ &= \left[\frac{2}{3a} (ax)^{\frac{3}{2}} - \frac{1}{3a} x^3 \right] \Big|_0^a \\ &= \frac{a^2}{3}. \end{aligned}$$

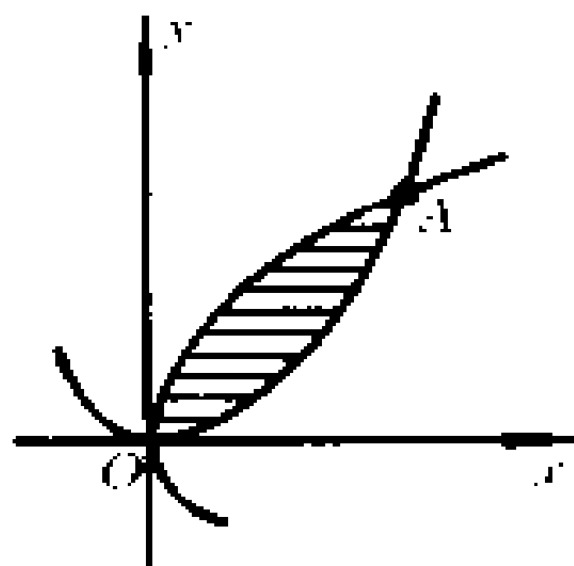


图 4.18

2398. $y = x^2, x + y = 2.$

解 如图 4.19 所示, 交点为 $A(-2, 4)$ 及 $B(1, 1)$.

所求的面积为

$$\begin{aligned} S &= \int_{-2}^1 [(2-x) - x^2] dx \\ &= \left(2x - \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_{-2}^1 \\ &= 4 \frac{1}{2}. \end{aligned}$$

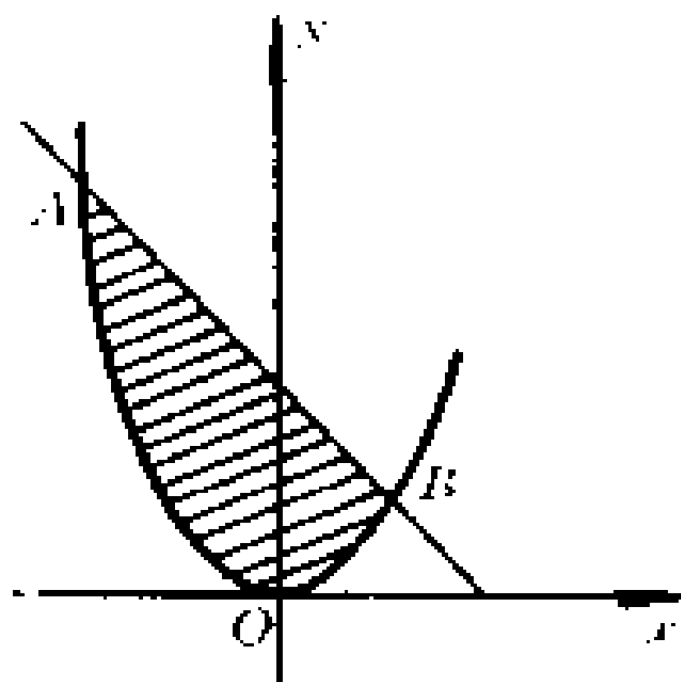


图 4.19

2399. $y = 2x - x^2, x + y = 0.$

解 如图 4.20 所示, 交点为 $A(3, -3)$ 及 $O(0, 0)$.

所求的面积为

$$\begin{aligned} S &= \int_0^3 [(2x - x^2) - (-x)] dx \\ &= \left(\frac{3x^2}{2} - \frac{1}{3}x^3 \right) \Big|_0^3 = 4 \frac{1}{2}. \end{aligned}$$

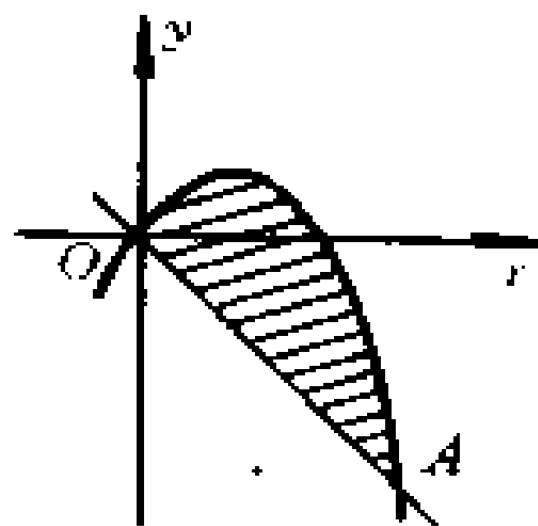


图 4.20

2400. $y = |\lg x|, y = 0, x = 0.1, x = 10.$

解 如图 4.21 所示, 所求的面积为

$$\begin{aligned}
S &= - \int_{0.1}^1 \lg x dx \\
&\quad + \int_1^{10} \lg x dx \\
&= (-x \lg x \\
&\quad + x \lg e) \Big|_{0.1}^1 \\
&\quad + (x \lg x \\
&\quad - x \lg e) \Big|_1^{10} \\
&= 9.9 - 8.1 \lg e \\
&\doteq 6.38.
\end{aligned}$$

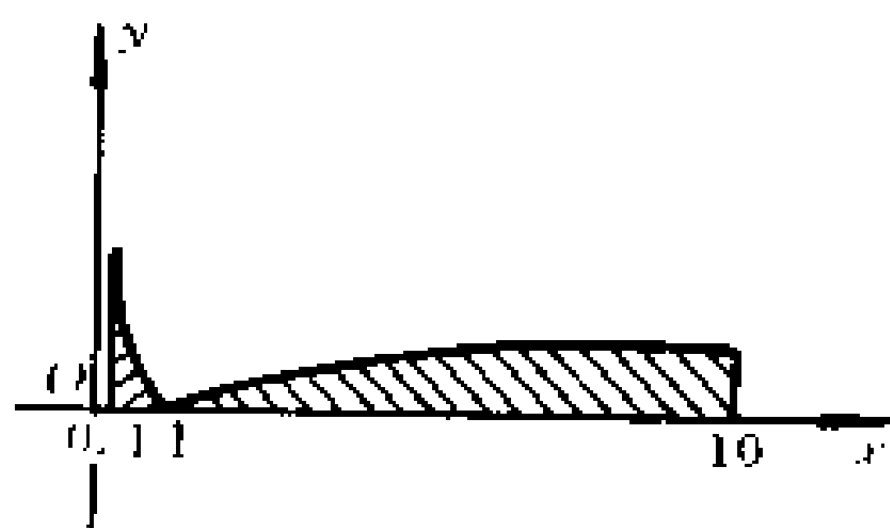


图 4.21

2401. $y = x, y = x + \sin^2 x [0 \leq x \leq \pi]$.

解 所求的面积为

$$\begin{aligned}
S &= \int_0^\pi (x + \sin^2 x - x) dx \\
&= \left(\frac{x}{2} - \frac{1}{4} \sin 2x \right) \Big|_0^\pi = \frac{\pi}{2}.
\end{aligned}$$

2402. $y = \frac{a^3}{a^2 + x^2}, y = 0$.

解 所求的面积为

$$\begin{aligned}
S &= \int_{-\infty}^{+\infty} \frac{a^3}{a^2 + x^2} dx = 2a^3 \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{a^2 + x^2} \\
&= 2a^3 \cdot \lim_{b \rightarrow +\infty} \frac{1}{a} \arctan \frac{x}{a} = \pi a^2.
\end{aligned}$$

2403. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

解 所求的面积为

$$S = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

2406. $Ax^2 + 2Bxy + Cy^2 = 1 (AC - B^2 > 0)$.

解 解此方程,得

$$y_1 = \frac{-Bx - \sqrt{B^2x^2 - C(Ax^2 - 1)}}{C},$$

及

$$y_2 = \frac{-Bx + \sqrt{B^2x^2 - C(Ax^2 - 1)}}{C}.$$

当 $B^2x^2 - C(Ax^2 - 1) \geq 0$, 即 $|x| \leq \sqrt{\frac{C}{AC - B^2}}$ 时,
 y_1 及 y_2 才有实数值.

设

$$a = \sqrt{\frac{C}{AC - B^2}}$$

则所求的面积为

$$\begin{aligned} S &= \int_{-a}^a (y_2 - y_1) dx \\ &= \frac{2}{C} \int_{-a}^a \sqrt{C^2 - (AC - B^2)x^2} dx \\ &= \frac{2}{C} \sqrt{AC - B^2} \int_{-a}^a \sqrt{a^2 - x^2} dx \\ &= \frac{2}{C} \sqrt{AC - B^2} \cdot \frac{\pi}{2} a^2 = \frac{\pi}{\sqrt{AC - B^2}}. \end{aligned}$$

2407. $y^2 = \frac{x^3}{2a - x}$ (蔓叶线), $x = 2a$.

解 所求的面积为

$$\begin{aligned} S &= 2 \int_0^{2a} x \sqrt{\frac{x}{2a - x}} dx \\ &= 16a^2 \int_0^{+\infty} \frac{t^4}{(t^2 + 1)^3} dt, \end{aligned}$$

$$\begin{aligned}
&= 16a^2 \lim_{b \rightarrow +\infty} \int_0^b \left(\frac{1}{t^2 + 1} - \frac{2}{(t^2 + 1)^2} \right. \\
&\quad \left. + \frac{1}{(t^2 + 1)^3} \right) dt \\
&= 16a^2 \lim_{b \rightarrow +\infty} \left[\frac{3}{8} \arctan t - \frac{5t}{8(t^2 + 1)} \right. \\
&\quad \left. + \frac{t}{4(t^2 + 1)^2} \right] \Big|_0^b \\
&= 3\pi a^2.
\end{aligned}$$

*) 设 $t = \sqrt{\frac{x}{2a - x}}$.

**) 利用 1921 题的递推公式.

2408. $x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y}$
 $- \sqrt{a^2 - y^2}$ (曳物
 线),

$y = 0$.

解 如图 4.24 所示,

所求的面积为

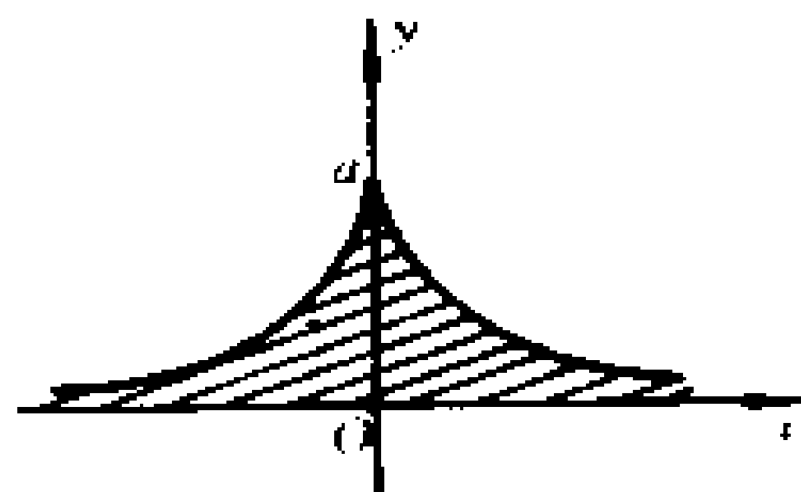


图 4.24

$$\begin{aligned}
S &= 2 \int_0^a \left(a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \right) dy \\
&= 2a \lim_{\epsilon \rightarrow +0} \int_{\epsilon}^a \ln \frac{a + \sqrt{a^2 - y^2}}{y} dy \\
&\quad - 2 \left(\frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \arcsin \frac{y}{a} \right) \Big|_0^a \\
&= 2a \lim_{\epsilon \rightarrow +0} \left(y \ln \frac{a + \sqrt{a^2 - y^2}}{y} \right.
\end{aligned}$$

$$+ a \arcsin \frac{y}{a} \Big|_c^a - \frac{\pi a^2}{2}$$

$$= \pi a^2 - \frac{\pi a^2}{2} = \frac{\pi a^2}{2}.$$

2409. $y^2 = \frac{x^n}{(1+x^{n+2})^2} (x > 0; n > -2).$

解 所求的面积为

$$S = 2 \int_0^{+\infty} \frac{x^{\frac{n}{2}}}{1+x^{n+2}} dx$$

$$= 2 \lim_{\substack{c \rightarrow +0 \\ b \rightarrow +\infty}} \int_c^b \frac{2}{n+2} \cdot \frac{dt}{1+t^2}^{**}$$

$$= 2 \cdot \frac{2}{n+2} \lim_{b \rightarrow +\infty} \arctan t \Big|_0^b = \frac{2\pi}{n+2}.$$

*) 设 $t = x^{\frac{n+2}{2}}$.

2410. $y = e^{-x} \sin x, y = 0 (x \geq 0).$

解 令 $\sin x = 0$, 得 $x = k\pi (k = 0, \pm 1, \pm 2, \dots)$. 当 $x \geq 0$ 时, 由于 $\sin x$ 在 $(\pi, 2\pi), (3\pi, 4\pi), \dots, ((2k+1)\pi, 2k\pi), \dots$ 中的值为负, 而在 $(0, \pi), (2\pi, 3\pi), \dots, (2k\pi, (2k+1)\pi), \dots$ 中的值为正, 故所求的面积为

$$S = \int_0^\pi e^{-x} \sin x dx - \int_\pi^{2\pi} e^{-x} \sin x dx$$

$$+ \int_{2\pi}^{3\pi} e^{-x} \sin x dx - \dots$$

$$+ (-1)^k \int_{k\pi}^{(k+1)\pi} e^{-x} \sin x dx + \dots$$

$$= \lim_{n \rightarrow +\infty} \sum_{k=0}^n (-1)^k \int_{k\pi}^{(k+1)\pi} e^{-x} \sin x dx$$

$$= \lim_{n \rightarrow +\infty} \sum_{k=0}^n (-1)^k \cdot \frac{-e^{-x}(\sin x + \cos x)}{2} \Big|_{k\pi}^{(k+1)\pi}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^{k+1} \cdot \frac{1}{2} [e^{-(k+1)\pi} \cos(k+1)\pi \\
&\quad - e^{-k\pi} \cos k\pi] \\
&= \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{2} \cdot (-1)^{k+1} [(-1)^{k+1} e^{-(k+1)\pi} \\
&\quad - (-1)^k e^{-k\pi}] \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{k=0}^n [e^{-(k+1)\pi} + e^{-k\pi}] \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left[1 + 2e^{-\pi} \sum_{k=0}^{n-1} e^{-k\pi} + e^{-(n+1)\pi} \right] \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left[1 + 2e^{-\pi} \cdot \frac{1 - e^{-n\pi}}{1 - e^{-\pi}} + e^{-(n+1)\pi} \right] \\
&= \frac{1}{2} \left(1 + \frac{2e^{-\pi}}{1 - e^{-\pi}} \right) = \frac{1}{2} \cdot \frac{e^{\pi} + 1}{e^{\pi} - 1} \\
&= \frac{1}{2} \operatorname{cth} \frac{\pi}{2} \doteq 0.545.
\end{aligned}$$

2411. 抛物线 $y^2 = 2x$ 分圆 $x^2 + y^2 = 8$ 的面积为两部分, 这两部分的比如何?

解 抛物线 $y^2 = 2px$ 和圆 $x^2 + y^2 = 8$ 在第一象限内的交点为 $A(2, 2)$.

设这两部分的面积分别为 S_1 及 S_2 (图 4.25), 则有

$$S_1 = 2 \int_0^2 \left(\sqrt{8 - y^2} - \frac{y^2}{2} \right) dy$$

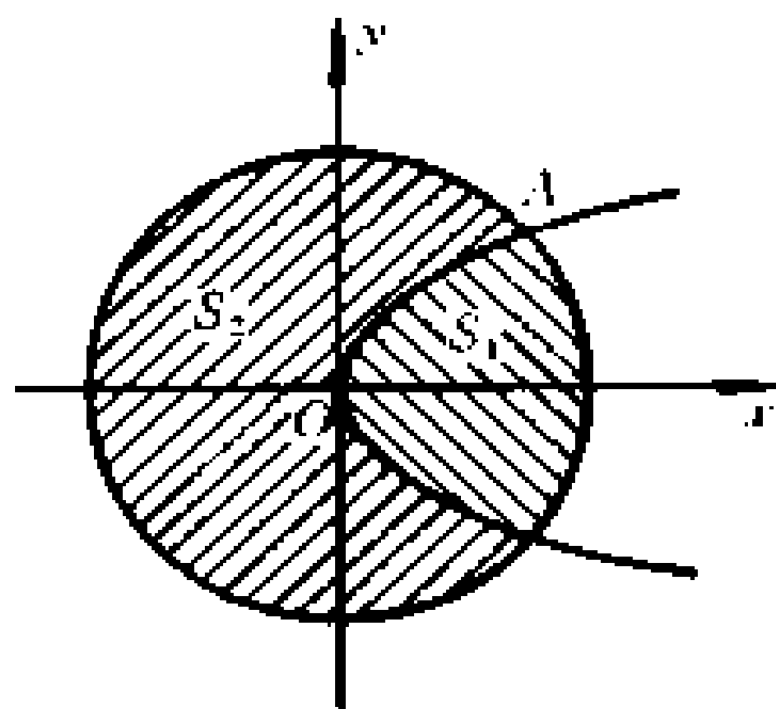


图 4.25

$$\begin{aligned}
&= 2 \left(\frac{y}{2} \sqrt{8 - y^2} + \frac{8}{2} \arcsin \frac{y}{2\sqrt{2}} - \frac{1}{6} y^3 \right) \Big|_0^2 \\
&= 2\pi + \frac{4}{3},
\end{aligned}$$

及

$$S_2 = 8\pi - \left(2\pi + \frac{4}{3} \right) = 6\pi - \frac{4}{3}.$$

于是, 它们之比为

$$\frac{S_1}{S_2} = \frac{2\pi + \frac{4}{3}}{6\pi - \frac{4}{3}} = \frac{3\pi + 2}{9\pi - 2}.$$

2412. 把双曲线 $x^2 - y^2 = a$ 上的点 $M(x, y)$ 的坐标表成为双曲线扇形 $S = OM'M$ 面积的函数. 这个扇形是由双曲线的弧 $M'M$ 与二射线 OM 及 OM' 所围成, 其中 $M'(x, -y)$ 是对于 Ox 轴与 M 对称的点.

解 如图 4.26 所示, 则有

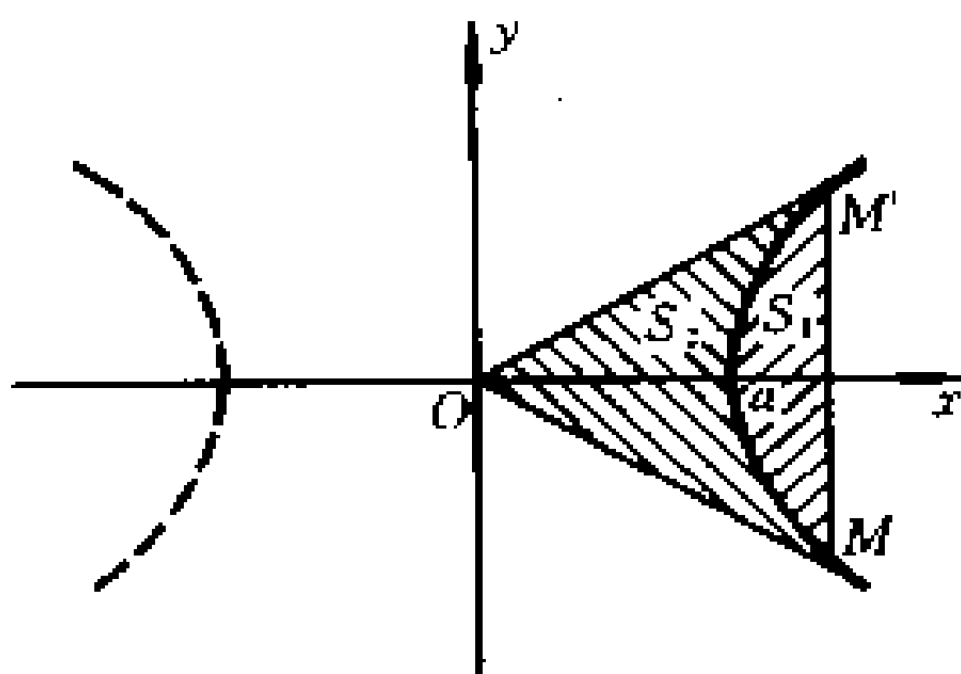


图 4.26

$$\frac{S_1}{2} = \int_a^x \sqrt{x^2 - a^2} dx$$

$$= \left[\frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2 - a^2}) \right] \Big|_a^x$$

$$= \frac{1}{2}xy - \frac{a^2}{2} \ln \frac{x+y}{a}$$

及

$$S_2 = 2 \left(\frac{xy}{2} - \frac{S_1}{2} \right) = a^2 \ln \frac{x+y}{a}.$$

若记 $S_2 = S$, 则由上式得

$$x + y = ae^{\frac{S}{a^2}}. \quad (1)$$

以(1)式代入 $x^2 - y^2 = a^2$ 中, 易得

$$x - y = ae^{-\frac{S}{a^2}}. \quad (2)$$

由(1)式及(2)式, 解得

$$x = a \cdot \frac{e^{\frac{S}{a^2}} + e^{-\frac{S}{a^2}}}{2} = a \operatorname{ch} \frac{S}{a^2}$$

及

$$y = a \cdot \frac{e^{\frac{S}{a^2}} - e^{-\frac{S}{a^2}}}{2} = a \operatorname{sh} \frac{S}{a^2}.$$

求由下列参数方程式所表曲线围成的面积:

2413. $x = a(t - \sin t), y = a(1 - \cos t) [0 \leq t \leq 2\pi]$ (摆线) 及 $y = 0$.

解 所求的面积为

$$S = \int_0^{2\pi} a(1 - \cos t) \cdot a(1 - \cos t) dt$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2} \right) dt$$

$$= a^2 \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t \right) \Big|_0^{2\pi} = 3\pi a^2.$$

由此可见,所求摆线一拱的面积等于原来母圆面积的三倍.

2414. $x = 2t - t^2, y = 2t^2 - t^3.$

解 当 $t = 0$ 及 2 时, $x = 0, y = 0$;

当 $0 < t < 2$ 时,

$$x > 0,$$

$$y > 0;$$

当 $t < 0$ 时,

$$x < 0,$$

$$y > 0;$$

当 $t > 2$ 时,

$$x < 0,$$

$$y < 0.$$

如图 4.27 所示,所求的面积为

$$\begin{aligned} S &= - \int_0^2 (2t^2 - t^3) \cdot 2(1 - t) dt \\ &= - 2 \int_0^2 (t^4 - 3t^3 + 2t^2) dt \\ &= \frac{8}{15}. \end{aligned}$$

2415. $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t) [0 \leq t \leq 2\pi]$ (圆的渐伸线) 及 $x = a, y \leq 0$.

解 所求面积为

$$S = - \int_0^{2\pi} a(\sin t - t \cos t) \cdot a t \cos t dt$$

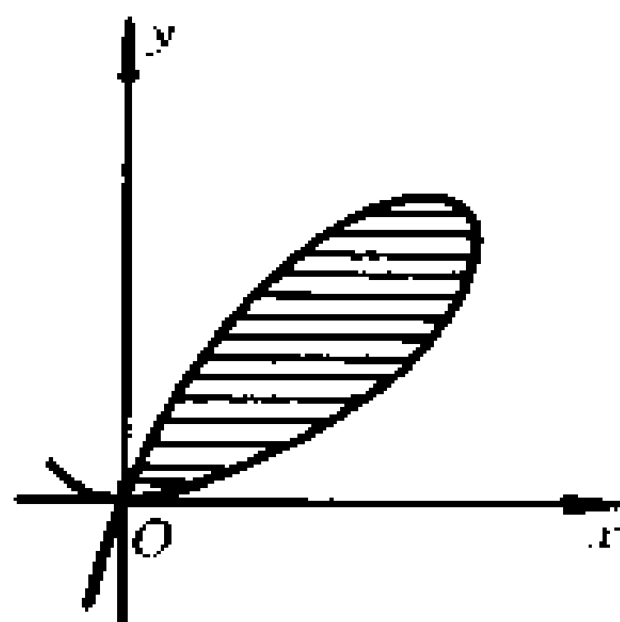


图 4.27

$$\begin{aligned}
& - \int_{AB} y dx \\
& = a^2 \left(\frac{1}{6} t^3 + \frac{1}{4} t^2 \sin 2t + \frac{1}{2} t \cos 2t \right. \\
& \quad \left. - \frac{1}{4} \sin 2t \right) \Big|_0^{2\pi} - \int_{AB} y dx \\
& = \frac{a^2}{3} (4\pi^3 + 3\pi) - \int_{AB} y dx,
\end{aligned}$$

其中 $\int_{AB} y dx$ 表沿着从点 $A(a, -2\pi a)$ 到点 $B(a, 0)$ 的直线 \overline{AB} 上的积分. 由于在 \overline{AB} 上 $x \equiv a$, 故 $dx = 0$, 从而 $\int_{AB} y dx = 0$. 于是, 得

$$S = \frac{a^2}{3} (4\pi^3 + 3\pi).$$

2416. $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$.

解 所求面积为

$$\begin{aligned}
S &= \frac{1}{2} \int_0^{2\pi} (xy'_t - yx'_t) dt \\
&= \frac{1}{2} \int_0^{2\pi} [a(2\cos t - \cos 2t) \cdot a(2\cos t \\
&\quad - 2\cos 2t) - a(2\sin t - \sin 2t) \\
&\quad \cdot a(-2\sin t + 2\sin 2t)] dt \\
&= 3a^2 \int_0^{2\pi} (1 - \cos t \cos 2t - \sin t \sin 2t) dt \\
&= 3a^2 \int_0^{2\pi} (1 - \cos t) dt = 6\pi a^2
\end{aligned}$$

2417. $x = \frac{c^2}{a} \cos^3 t$, $y = \frac{c^2}{b} \sin^3 t$ ($c^2 = a^2 - b^2$) (椭圆的渐屈线).

解 如图 4.28 所示, 所求的面积为

$$\begin{aligned}
 S &= 4 \int_0^{\frac{\pi}{2}} \frac{c^2}{b} \sin^3 t \\
 &\quad \cdot \frac{3c^2}{a} \cos^2 t \sin t dt \\
 &= \frac{12c^4}{ab} \int_0^{\frac{\pi}{2}} \sin^4 t \\
 &\quad (1 - \sin^2 t) dt \\
 &= \frac{3\pi c^4}{8ab}.
 \end{aligned}$$

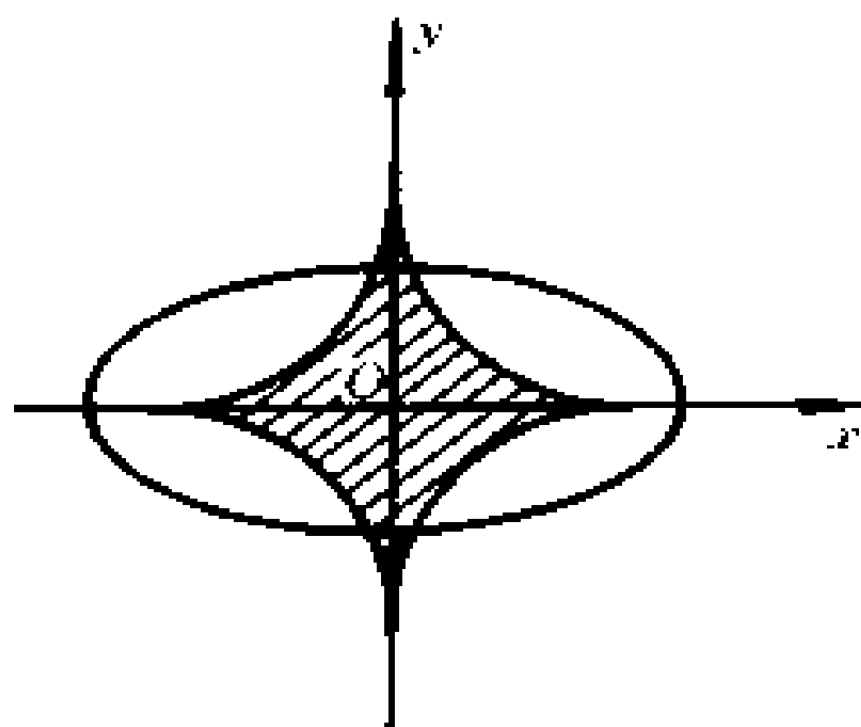


图 4.28

求由下列极坐标方程
式所表曲线围成的面
积 S ;

2418. $r^2 = a^2 \cos 2\varphi$
(双纽线).

解 如图 4.29 所示, 所求的面积为

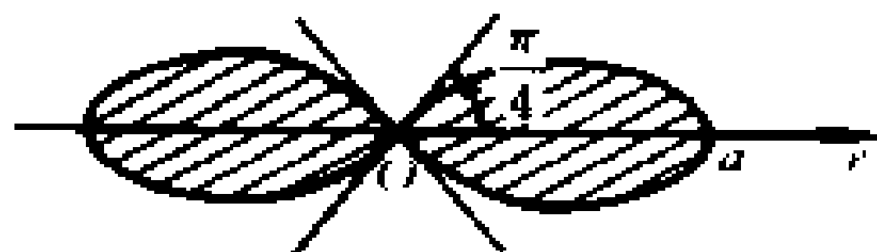


图 4.29

$$\begin{aligned}
 S &= 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \cos 2\varphi d\varphi \\
 &= a^2.
 \end{aligned}$$

2419. $r = a(1 + \cos\varphi)$ (心脏形线).

解 如图 4.30 所示, 所求的面积为

$$S = 2 \cdot \frac{1}{2} \int_0^\pi a^2 (1 + \cos\varphi)^2 d\varphi = \frac{3}{2} \pi a^2.$$

$$S = 2 \cdot \frac{1}{2} \int_0^\pi \frac{p^2 d\varphi}{(1 + \epsilon \cos \varphi)^2}$$

$$= p^2 \int_0^\pi \frac{d\varphi}{(1 + \epsilon \cos \varphi)^2}.$$

设

$$\operatorname{tg} \frac{\varphi}{2} = t,$$

并记

$$a^2 = \frac{1 + \epsilon}{1 - \epsilon},$$

则有

$$\int \frac{d\varphi}{(1 + \epsilon \cos \varphi)^2} = \int \frac{2(t^2 + 1)dt}{(1 - \epsilon)^2(t^2 + a^2)^2}$$

$$= \frac{2}{(1 - \epsilon)^2} \int \frac{dt}{t^2 + a^2}$$

$$+ \frac{2(1 - a^2)}{(1 - \epsilon)^2} \int \frac{dt}{(t^2 + a^2)^2}$$

$$= \frac{2}{a(1 - \epsilon)^2} \operatorname{arc} \operatorname{tg} \frac{t}{a}$$

$$+ \frac{2(1 - a^2)}{(1 - \epsilon)^2} \left\{ \frac{t}{2a^2(t^2 + a^2)} \right.$$

$$\left. + \frac{1}{2a^3} \operatorname{arc} \operatorname{tg} \frac{t}{a} \right\}^{*)} + C.$$

当 $0 \leq \varphi \leq \pi$ 时, $0 \leq t < +\infty$, 从而得一广义积分. 于是, 经计算得

$$S = \left\{ \frac{\pi}{a(1 - \epsilon)^2} + \frac{(1 - a^2)\pi}{2a^3(1 - \epsilon)^2} \right\} \cdot p^2$$

$$= \frac{\pi p^2}{(1 - \epsilon^2)^{\frac{3}{2}}}.$$

*) 利用 1921 题的递推公式.

$$2423. \quad r = a \cos \varphi, r = a(\cos \varphi + \sin \varphi) \left(M\left(\frac{a}{2}, 0\right) \in S \right).$$

解 如图 4.32 所示,

$$|OA| = a,$$

$$a = -\frac{\pi}{4},$$

阴影部分即为所求的面积.

曲线 $L_1: r = a \cos \varphi$,

$L_2: r = a(\cos \varphi + \sin \varphi)$.

所求的面积为

$$\begin{aligned} S &= \frac{\pi}{2} \left(\frac{a}{2} \right)^2 + \frac{1}{2} \int_{-\frac{\pi}{4}}^0 a^2 (\cos \varphi + \sin \varphi)^2 d\varphi \\ &= \frac{a^2(\pi - 1)}{4}. \end{aligned}$$

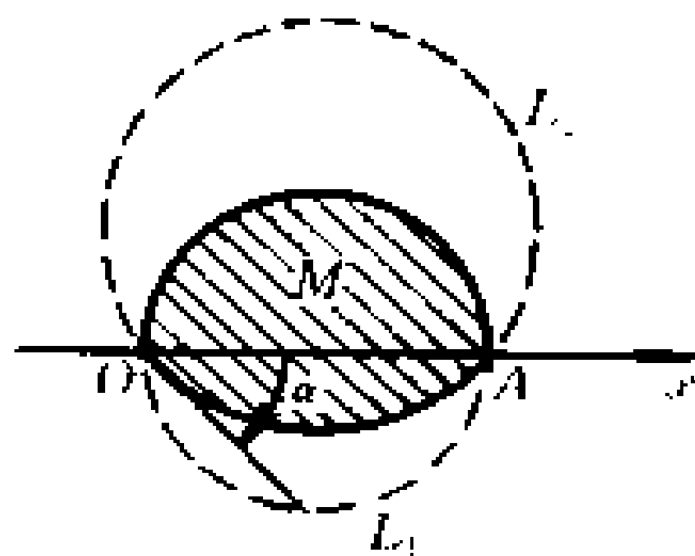


图 4.32

2424⁺. 求由曲线 $\varphi = r \arctgr r$ 及二射线 $\varphi = 0$ 及 $\varphi = \frac{\pi}{\sqrt{3}}$ 所围成之扇形的面积.

解 当 φ 由 0 变到 $\frac{\pi}{\sqrt{3}}$, r 从 0 变到 $\sqrt{3}$, 而

$$d\varphi = \left(\frac{r}{1+r^2} + \arctgr r \right) dr.$$

所求的面积为

$$\begin{aligned} S &= \frac{1}{2} \int_0^{\sqrt{3}} r^2 d\varphi \\ &= \frac{1}{2} \int_0^{\sqrt{3}} \left(\frac{r^3}{1+r^2} + r^2 \arctgr r \right) dr \end{aligned}$$

$$= \left[\frac{1}{6}r^2 - \frac{1}{6}\ln(1+r^2) + \frac{1}{6}r^3 \operatorname{arc} \operatorname{tgr} \right] \Big|_0^{\sqrt{3}}$$

$$= \frac{1}{2} - \frac{1}{3}\ln 2 + \frac{\sqrt{3}}{6}\pi.$$

2425. 求封闭曲线

$$r = \frac{2at}{1+t^2}, \varphi = \frac{\pi t}{1+t}$$

所包围的面积.

解 当曲线封闭时, t 由 0 变化到 $+\infty$. 所求的面积为

$$\begin{aligned} S &= \frac{1}{2} \int_0^{+\infty} r^2 d\varphi = 2\pi a^2 \int_0^{+\infty} \frac{t^2}{(1+t^2)^2(1+t)^2} dt \\ &= 2\pi a^2 \lim_{b \rightarrow +\infty} \left\{ \int_0^b \frac{dt}{4(1+t)^2} - \frac{1}{4} \int_0^b \frac{dt}{1+t^2} \right. \\ &\quad \left. + \frac{1}{2} \int_0^b \frac{t dt}{(1+t^2)^2} \right\} \\ &= 2\pi a^2 \lim_{b \rightarrow +\infty} \left\{ -\frac{1}{4(1+t)} - \frac{1}{4} \operatorname{arc} \operatorname{tgt} \right. \\ &\quad \left. - \frac{1}{4} \cdot \frac{1}{1+t^2} \right\} \Big|_0^b \\ &= \pi a^2 \left(1 - \frac{\pi}{4} \right). \end{aligned}$$

变为极坐标, 以求下列曲线所围成的面积:

2426. $x^3 + y^3 = 3axy$ (笛卡尔叶形线).

解 $r^3(\cos^3\varphi + \sin^3\varphi) = 3ar^2\cos\varphi\sin\varphi$,

于是

$$r = \frac{3a\sin\varphi\cos\varphi}{\sin^3\varphi + \cos^3\varphi}.$$

当 $\varphi \in \left[0, \frac{\pi}{2}\right]$ 时, $r \geq 0$, 且当 $\varphi = 0$ 及 $\varphi = \frac{\pi}{2}$ 时, $r =$

0. 所以, 从 $\varphi = 0$ 到 $\varphi = \frac{\pi}{2}$, 叶形线位于第一象限部分所围成的面积, 即为所要求的面积 (图 4.33)

$$\begin{aligned} S &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{9a^2 \sin^2 \varphi \cos^2 \varphi}{(\sin^3 \varphi + \cos^3 \varphi)^2} d\varphi \\ &= \frac{9a^2}{2} \int_0^{+\infty} \frac{t^2 dt}{(1+t^3)^2} \quad *) \\ &= \frac{9a^2}{2} \lim_{b \rightarrow +\infty} \left. \frac{-1}{3(1+t^3)} \right|_0^b \\ &= \frac{3a^2}{2}. \end{aligned}$$

*) 设 $\operatorname{tg} \varphi = t$.

2427. $x^4 + y^4 = a^2(x^2 + y^2)$.

解 $r^4(\sin^4 \varphi + \cos^4 \varphi)$
 $= a^2 r^2,$

于是

$$r = \frac{\sqrt{2} a}{\sqrt{2 - \sin^2 2\varphi}}.$$

如图 4.34 所示, 所求的面积为

$$\begin{aligned} S &= 8 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{2a^2}{2 - \sin^2 2\varphi} d\varphi \\ &= 4a^2 \int_0^{\frac{\pi}{2}} \frac{1}{2 - \sin^2 t} dt \\ &= \frac{2a^2}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2} - \sin t} + \frac{1}{\sqrt{2} + \sin t} \right) dt \\ &= \sqrt{2} a^2 \left\{ 2 \operatorname{arc} \operatorname{tg} \left(\sqrt{2} \operatorname{tg} \frac{t}{2} - 1 \right) \right. \\ &\quad \left. + 2 \operatorname{arc} \operatorname{tg} \left(\sqrt{2} \operatorname{tg} \frac{t}{2} + 1 \right) \right\} \Big|_0^{\frac{\pi}{2}} \end{aligned}$$

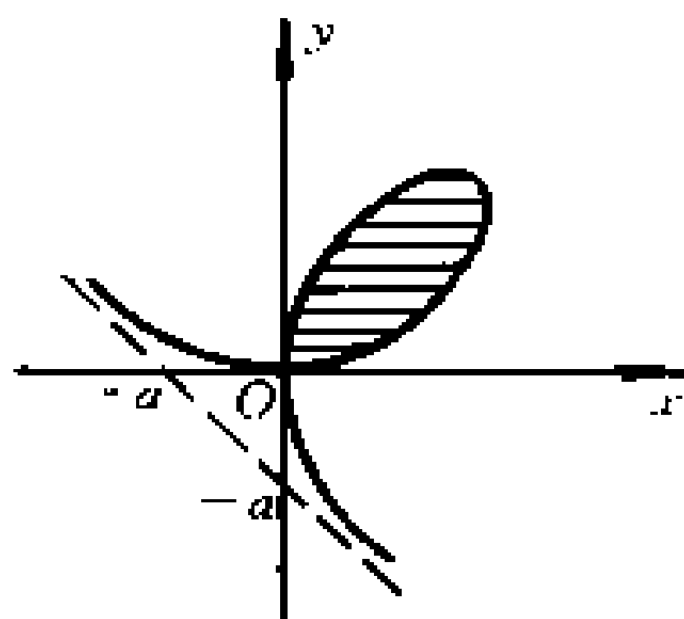


图 4.33

$$\begin{aligned}
&= 2\sqrt{2}a^2\{\arctan(\sqrt{2}-1) + \arctan(\sqrt{2}+1)\} \\
&= 2\sqrt{2}a^2 \cdot \frac{\pi}{2} = \sqrt{2}\pi a^2.
\end{aligned}$$

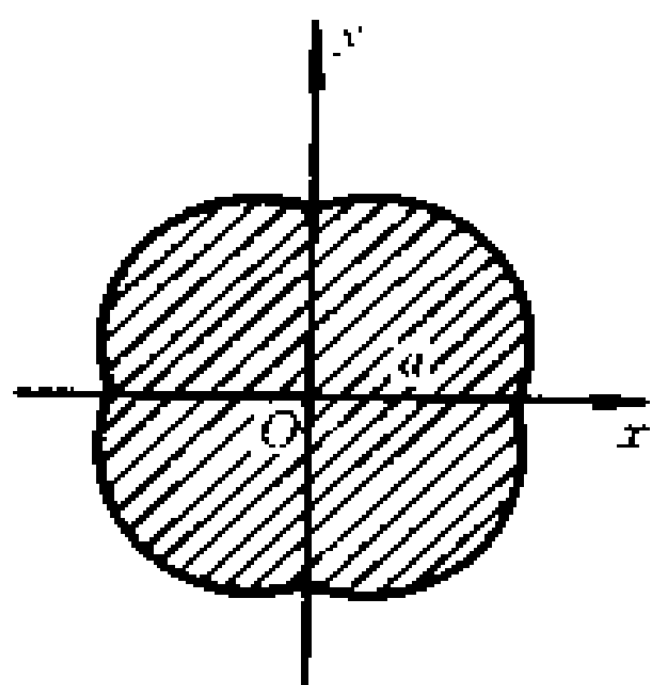


图 4.34

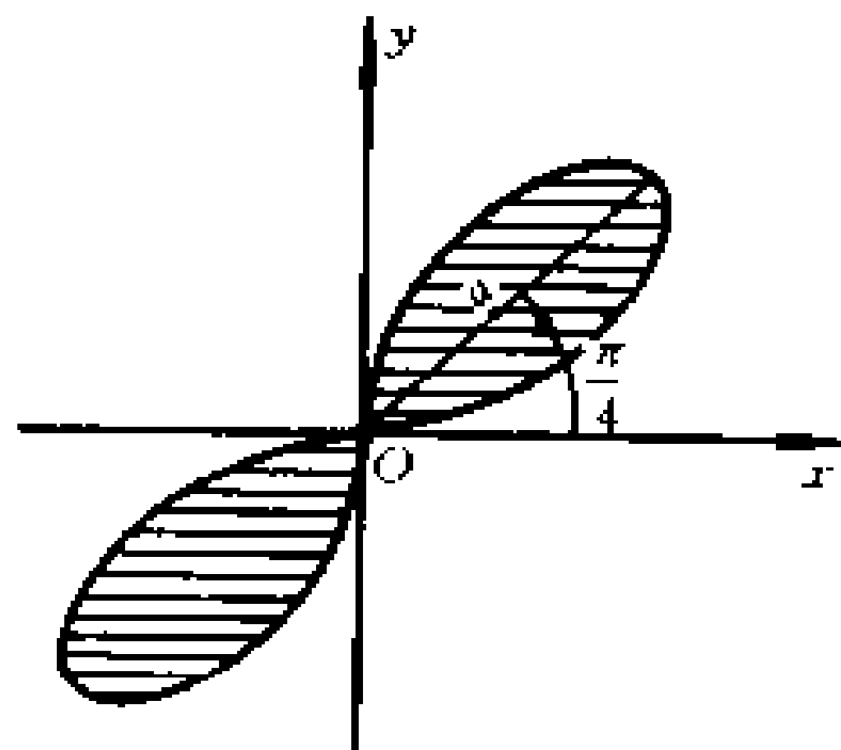


图 4.35

2428. $(x^2 + y^2)^2 = 2a^2xy$ (双纽线).

解 $r^2 = a^2 \sin 2\varphi$ (图 4.35).

所求的面积为

$$S = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \sin 2\varphi = a^2.$$

化方程式为参数式的形状,以求下列曲线所围成的面积:

2429. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (内摆线).

解 设

$$x = a \cos^3 t, y = a \sin^3 t,$$

其中 $0 \leq t \leq \frac{\pi}{2}$, 它对应于四分之一的面积. 所求的面积为其四倍, 即

$$\begin{aligned}
 S &= 4 \int_0^a y \, dx = 4 \int_{\frac{\pi}{2}}^0 (-3a^2 \sin^4 t \cos^2 t) dt \\
 &= 12a^2 \int_0^{\frac{\pi}{2}} (\sin^4 t - \sin^6 t) dt = \frac{3\pi a^2}{8}.
 \end{aligned}$$

2430. $x^4 + y^4 = ax^2y$.

解 设

$$y = tx,$$

则曲线的参数方程为

$$\begin{cases} x = \frac{at}{1+t^4}, \\ y = \frac{at^2}{1+t^4}. \end{cases} \quad (-\infty < t < +\infty)$$

利用对称性知, 所求的面积为

$$\begin{aligned}
 S &= -2 \int_0^{+\infty} \frac{at^2}{1+t^4} \cdot \frac{a(1-3t^4)}{(1+t^4)^2} dt \\
 &= -2a^2 \left(\int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt \right. \\
 &\quad \left. - 3 \int_0^{+\infty} \frac{t^6}{(1+t^4)^3} dt \right).
 \end{aligned}$$

因为

$$\begin{aligned}
 &\int \frac{x^n dx}{(a+bx^4)^m} \\
 &= \frac{x^{n-3}}{(n+1-4m)b \cdot (a+bx^4)^{m-1}} \\
 &\quad - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4}}{(a+bx^4)^m} dx,
 \end{aligned}$$

所以

$$\int_0^{+\infty} \frac{t^6}{(1+t^4)^3} dt$$

$$\begin{aligned}
&= -\frac{t^3}{5(1+t^4)^2} \Big|_0^{+\infty} + \frac{3}{5} \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt \\
&= \frac{3}{5} \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt,
\end{aligned}$$

于是

$$S = \frac{8}{5} a^2 \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt.$$

又因

$$\begin{aligned}
&\int \frac{x^n dx}{(a+bx^4)^m} \\
&= \frac{x^{n+1}}{4a(m-1)(a+bx^4)^{m-1}} \\
&\quad + \frac{4m-n-5}{4a(m-1)} \int \frac{x^n dx}{(a+bx^4)^{m-1}}, \dots
\end{aligned}$$

所以

$$\begin{aligned}
&\int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt \\
&= \frac{t^3}{8(1+t^4)^2} \Big|_0^{+\infty} + \frac{5}{8} \int_0^{+\infty} \frac{t^2}{(1+t^4)^2} dt \\
&= \frac{5}{8} \int_0^{+\infty} \frac{t^2}{(1+t^4)^2} dt \\
&= \frac{5}{8} \left[\frac{t^3}{4(1+t^4)} \Big|_0^{+\infty} + \frac{1}{4} \int_0^{+\infty} \frac{t^2 dt}{1+t^4} \right] \\
&= \frac{5}{32} \int_0^{+\infty} \frac{t^2}{1+t^4} dt,
\end{aligned}$$

于是

$$S = \frac{1}{4} a^2 \int_0^{+\infty} \frac{t^2}{1+t^4} dt.$$

利用

$$\int \frac{x^2}{a+bx^4} dx$$

$$= \frac{1}{4b \cdot \sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}} \left\{ \ln \frac{x^2 - \sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}x + \sqrt{\frac{a}{b}}}{x^2 + \sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}x + \sqrt{\frac{a}{b}}} \right. \\ \left. + 2 \operatorname{arc} \operatorname{tg} \frac{\sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}x}{\sqrt{\frac{a}{b}} - x^2} \right\} \quad (ab > 0),$$

即得

$$\int \frac{t^2}{1+t^4} dt \\ = \frac{1}{4\sqrt{2}} \left\{ \ln \frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} \right. \\ \left. + 2 \operatorname{arc} \operatorname{tg} \frac{\sqrt{2}t}{1-t^2} \right\} + C.$$

考虑到上述式子右端的函数 $\operatorname{arc} \operatorname{tg} \frac{\sqrt{2}t}{1-t^2}$ 在 $(0, +\infty)$ 中的 $t=1$ 点间断, 并且

$$\lim_{t \rightarrow 1+0} \operatorname{arc} \operatorname{tg} \frac{\sqrt{2}t}{1-t^2} = -\frac{\pi}{2},$$

及

$$\lim_{t \rightarrow 1-0} \operatorname{arc} \operatorname{tg} \frac{\sqrt{2}t}{1-t^2} = \frac{\pi}{2},$$

于是

$$\int_0^{+\infty} \frac{t^2}{1+t^4} dt = \int_0^1 \frac{t^2}{1+t^4} dt + \int_1^{+\infty} \frac{t^2}{1+t^4} dt \\ = \frac{1}{4\sqrt{2}} \left\{ \ln \frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} + 2 \operatorname{arc} \operatorname{tg} \frac{\sqrt{2}t}{1-t^2} \right\} \Big|_0^1$$

$$\begin{aligned}
& + \frac{1}{4\sqrt{2}} \left\{ \ln \frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} \right. \\
& \left. + 2 \operatorname{arc\,tg} \frac{\sqrt{2}t}{1-t^2} \right\} \Big|_1^{+\infty} \\
& = \frac{2}{4\sqrt{2}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\sqrt{2}\pi}{4},
\end{aligned}$$

最后得所求的面积为

$$S = \frac{\sqrt{2}\pi}{16} a^2.$$

*) 参阅“函数表与积分表”(И. М. 雷日克, И. С. 格拉德什坦) 第 64 页“(2.133)2”.

**) 参阅同书第 64 页“(2.133)1”.

***) 参阅同书第 64 页“(2.132)3”.

§ 6. 弧长的计算法

1° 在直角坐标系中的弧长 平滑(连续可微分的)曲线

$$y = y(x) (a \leq x \leq b)$$

上一段弧的长度等于

$$s = \int_a^b \sqrt{1 + y'^2(x)} dx.$$

2° 参数方程所表曲线的弧长 若曲线 C 用参数方程式给出

$$x = x(t), y = y(t) (t_0 \leq t \leq T),$$

式中 $x(t), y(t)$ 为在闭区间 $[t_0, T]$ 内可微分的连续函数, 则曲线 C 的弧长等于

$$s = \int_{t_0}^T \sqrt{x'^2(t) + y'^2(t)} dt.$$

$$\begin{aligned}
& + \frac{p}{2\sqrt{2}} \ln \left[\sqrt{x} + \sqrt{x + \frac{p}{2}} \right] \Bigg|_0^{x_0} \\
& = 2\sqrt{x_0 \left(x_0 + \frac{p}{2} \right)} \\
& \quad + p \ln \left[\frac{\sqrt{x_0} + \sqrt{x_0 + \frac{p}{2}}}{\sqrt{\frac{p}{2}}} \right].
\end{aligned}$$

2433. $y = a \operatorname{ch} \frac{x}{a}$ 从点 $A(0, a)$ 至点 $B(b, h)$.

解 所求的弧长为

$$\begin{aligned}
s &= \int_0^b \sqrt{1 + \operatorname{sh}^2 \frac{x}{a}} dx = \int_0^b \operatorname{ch} \frac{x}{a} dx \\
&= a \operatorname{sh} \frac{x}{a} \Bigg|_0^b = a \operatorname{sh} \frac{b}{a} = \sqrt{h^2 - a^2}^{**}.
\end{aligned}$$

$$\begin{aligned}
*) \quad \text{由于 } h &= a \operatorname{ch} \frac{b}{a}, \text{ 故 } \operatorname{sh} \frac{b}{a} = \sqrt{\operatorname{ch}^2 \frac{b}{a} - 1} \\
&= \frac{1}{a} \sqrt{h^2 - a^2}
\end{aligned}$$

2434. $y = e^x (0 \leq x \leq x_0)$.

解 所求的弧长为

$$\begin{aligned}
s &= \int_0^{x_0} \sqrt{1 + e^{2x}} dx \\
&= \left[\sqrt{1 + e^{2x}} + \frac{1}{2} \ln \frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1} \right] \Bigg|_0^{x_0} \\
&= \sqrt{1 + e^{2x_0}} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{1 + e^{2x_0}} - 1}{\sqrt{1 + e^{2x_0}} + 1}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} \\
& = x_0 - \sqrt{2} + \sqrt{1+e^{2x_0}} \\
& - \ln\frac{1+\sqrt{1+e^{2x_0}}}{1+\sqrt{2}}.
\end{aligned}$$

2435. $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y (1 \leq y \leq e).$

解 所求的弧长为

$$\begin{aligned}
s &= \int_1^e \sqrt{1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2} dy \\
&= \int_1^e \frac{1+y^2}{2y} dy = \frac{e^2+1}{4}.
\end{aligned}$$

2436. $y = a \ln \frac{a^2}{a^2-x^2} (0 \leq x \leq b < a).$

解 $y' = \frac{2ax}{a^2-x^2}, \sqrt{1+y'^2} = \frac{a^2+x^2}{a^2-x^2}.$

所求的弧长为

$$s = \int_0^b \frac{a^2+x^2}{a^2-x^2} dx = a \ln \frac{a+b}{a-b} - b.$$

2437. $y = \ln \cos x \left(0 \leq x \leq a < \frac{\pi}{2} \right).$

解 所求的弧长为

$$\begin{aligned}
s &= \int_0^a \sqrt{1 + \operatorname{tg}^2 x} dx \\
&= \int_0^a \frac{dx}{\cos x} = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{a}{2} \right).
\end{aligned}$$

2438. $x = a \ln \frac{a + \sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2} (0 < b \leq y \leq a).$

解 $\frac{dx}{dy} = -\frac{\sqrt{a^2-y^2}}{y}, \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{a}{y}.$

所求的弧长为

$$s = \int_b^a \frac{a}{y} dy = a \ln \frac{a}{b}.$$

2439. $y^2 = \frac{x^3}{2a-x} \left(0 \leq x \leq \frac{5}{3}a \right) \quad **$.

解 如图 4.36 所示.

设 $y = tx$, 得

$$\begin{cases} x = \frac{2at^2}{1+t^2}, \\ y = \frac{2at^3}{1+t^2}. \end{cases}$$

当 $0 \leq x \leq \frac{5}{3}a$ 时, $0 \leq t \leq \sqrt{5}$ (一半弧长).

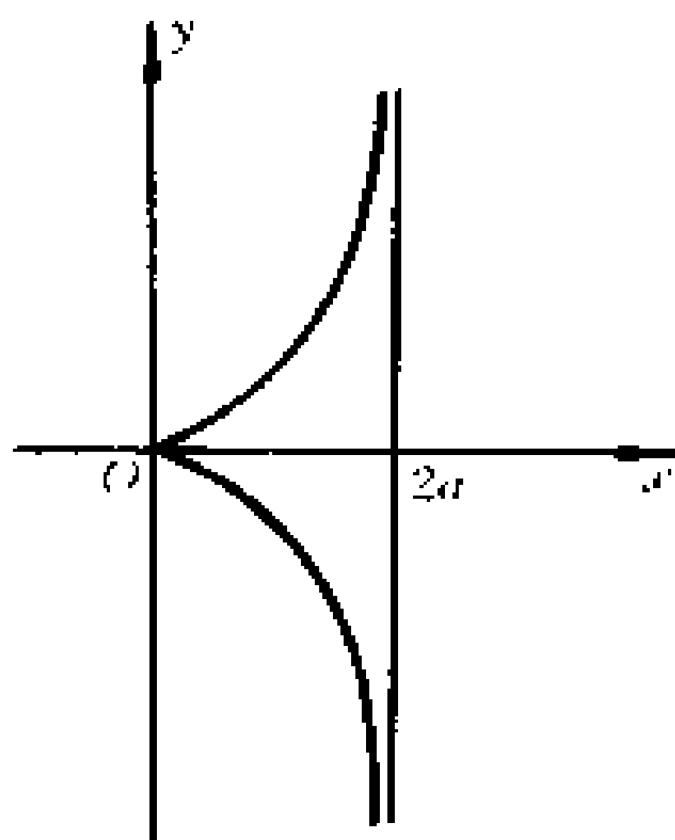


图 4.36

$$x_t' = \frac{4at}{(t^2+1)^2},$$

$$y_t' = \frac{2at^4 + 6at^2}{(t^2+1)^2},$$

$$\sqrt{x_t'^2 + y_t'^2} = \frac{2at \sqrt{t^2+4}}{t^2+1}.$$

所求的弧长为

$$\begin{aligned} s &= \int_0^{\sqrt{5}} 2 \frac{2at \sqrt{t^2+4}}{t^2+1} dt \\ &= 32a \cdot \int_0^{\arctg \frac{\sqrt{5}}{2}} \frac{\sin \theta d\theta}{\cos^2 \theta (1+3\sin^2 \theta)} \quad *** \\ &= \frac{32a}{3} \int_1^{\frac{2}{3}} \frac{dz}{z^2 \left(z^2 - \frac{4}{3} \right)} \quad **** \end{aligned}$$

$$= \frac{32a}{3} \left\{ \frac{3}{4} \cdot \frac{1}{z} + \frac{3\sqrt{3}}{16} \ln \frac{z - \frac{2}{\sqrt{3}}}{z + \frac{2}{\sqrt{3}}} \right\} \Big|_1^{\frac{2}{3}}$$

$$= 4a \left[1 + 3\sqrt{3} \ln \frac{1 + \sqrt{3}}{\sqrt{2}} \right].$$

*) 原题误为 $y^2 = \frac{x^2}{2a-x}$, 现按原答案予以改正.

* *) 设 $t = 2\operatorname{tg}\theta$.

* * *) 设 $z = \cos\theta$.

2440. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (内摆线).

解 $y' = -\sqrt[3]{\frac{y}{x}}, \sqrt{1+y'^2} = \left(\frac{a}{x}\right)^{\frac{1}{3}}.$

所求的弧长为

$$s = 4 \int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}} dx = 6a.$$

2441. $x = \frac{c^2}{a} \cos^3 t, y = \frac{c^2}{b} \sin^3 t, c^2 = a^2 - b^2$ (椭圆的渐屈线).

解 $\sqrt{x_t'^2 + y_t'^2}$
 $= \frac{3c^2}{ab} \sin t \cos t \sqrt{b^2 \cos^2 t + a^2 \sin^2 t}.$

所求的弧长为

$$s = 4 \int_0^{\frac{\pi}{2}} \frac{3c^2}{ab} \sin t \cos t \sqrt{b^2 \cos^2 t + a^2 \sin^2 t} dt$$

$$= \frac{12c^2}{3ab(a^2 - b^2)} \{b^2 + (a^2 - b^2) \sin^2 t\}^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{4(a^3 - b^3)}{ab}.$$

2442⁺. $x = a\cos^4 t, y = a\sin^4 t$.

解 $\sqrt{x_t'^2 + y_t'^2}$
 $= 4a\sin t \cos t \sqrt{\cos^4 t + \sin^4 t}.$

所求的弧长为

$$\begin{aligned} s &= \int_0^{\frac{\pi}{2}} 4a\sin t \cos t \sqrt{\cos^4 t + \sin^4 t} dt \\ &= 2a \int_0^{\frac{\pi}{2}} \sqrt{2\left(\sin^2 t - \frac{1}{2}\right)^2 + \frac{1}{2}} d\left(\sin^2 t - \frac{1}{2}\right) \\ &= 2a \left[\frac{\sin^2 t - \frac{1}{2}}{2} \sqrt{\cos^4 t + \sin^4 t} \right. \\ &\quad \left. + \frac{1}{4\sqrt{2}} \ln \left| \sin^2 t - \frac{1}{2} \right| \right. \\ &\quad \left. + \sqrt{\frac{1}{2}(\cos^4 t + \sin^4 t)} \right] \bigg|_0^{\frac{\pi}{2}} \\ &= \left(1 + \frac{1}{\sqrt{2}} \ln(1 + \sqrt{2}) \right) a. \end{aligned}$$

2443. $x = a(t - \sin t), y = a(1 - \cos t) (0 \leq t \leq 2\pi).$

解 所求的弧长为

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt \\ &= 2a \int_0^{2\pi} \sin \frac{t}{2} dt = 8a. \end{aligned}$$

2444. $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t) (0 \leq t \leq 2\pi)$ (圆的渐伸线).

解 $x_t' = a t \cos t, y_t' = a t \sin t,$

$$\sqrt{x_t'^2 + y_t'^2} = at.$$

所求的弧长为

$$s = \int_0^{2\pi} at \, dt = 2\pi^2 a.$$

2445⁺. $x = a(\operatorname{sh} t - t), y = a(\operatorname{ch} t - 1) (0 \leq t \leq T).$

解 $\sqrt{x_1'^2 + y_1'^2} = \sqrt{2}a \cdot \sqrt{\operatorname{ch}^2 t - \operatorname{ch} t}.$

所求的弧长为

$$\begin{aligned} s &= \int_0^T \sqrt{2}a \sqrt{\operatorname{ch}^2 t - \operatorname{ch} t} dt \\ &= \sqrt{2}a \int_1^{\operatorname{ch} T} \sqrt{\frac{\theta}{\theta+1}} d\theta \quad *) \\ &= 2\sqrt{2}a \int_{\frac{\pi}{4}}^{\operatorname{arc} \operatorname{tg} \sqrt{\operatorname{ch} T}} \frac{\sin^2 z}{\cos^3 z} dz \quad **) \\ &= 2\sqrt{2}a \left\{ \frac{\sin z}{2\cos^2 z} - \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{z}{2} \right) \right\} \Big|_{\frac{\pi}{4}}^{\operatorname{arc} \operatorname{tg} \sqrt{\operatorname{ch} T}} \\ &= \sqrt{2}a (\sqrt{\operatorname{ch} T} \cdot \sqrt{1 + \operatorname{ch} T} - \sqrt{2}) \\ &\quad - \sqrt{2}a [\ln(\sqrt{\operatorname{ch} T} + \sqrt{1 + \operatorname{ch} T}) \\ &\quad - \ln(1 + \sqrt{2})] \\ &= 2a \left(\operatorname{ch} \frac{T}{2} \cdot \sqrt{\operatorname{ch} T} - 1 \right) \\ &\quad - \sqrt{2}a \ln \frac{\sqrt{2} \operatorname{ch} \frac{T}{2} + \sqrt{\operatorname{ch} T}}{\sqrt{2} + 1} \quad ***) \end{aligned}$$

*) 设 $\theta = \operatorname{ch} t.$

**) 设 $\theta = \operatorname{tg}^2 z.$

***) $\sqrt{1 + \operatorname{ch} T} = \sqrt{2} \operatorname{ch} \frac{T}{2}.$

2446. $r = a\varphi$ (阿基米得螺旋线) $(0 \leq \varphi \leq 2\pi).$

解 所求的弧长为

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{a^2 \varphi^2 + a^2} d\varphi \\
&= a \left\{ \frac{\varphi}{2} \sqrt{\varphi^2 + 1} + \frac{1}{2} \ln(\varphi + \sqrt{\varphi^2 + 1}) \right\} \Big|_0^{2\pi} \\
&= a \left\{ \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln(2\pi + \sqrt{1 + 4\pi^2}) \right\}.
\end{aligned}$$

2447. $r = ae^{m\varphi}$ ($m > 0$) 当 $0 < r < a$.

解 $0 < r < a$, $-\infty < \varphi < 0$.

所求的弧长为

$$\begin{aligned}
s &= \int_{-\infty}^0 \sqrt{a^2 e^{2m\varphi} + a^2 m^2 e^{2m\varphi}} d\varphi \\
&= a \sqrt{m^2 + 1} \int_{-\infty}^0 e^{m\varphi} d\varphi \\
&= \lim_{a \rightarrow -\infty} \int_a^0 e^{m\varphi} d\varphi = \frac{a \sqrt{1 + m^2}}{m}.
\end{aligned}$$

2448. $r = a(1 + \cos\varphi)$.

解 $\sqrt{r^2 + r'^2} = 2a \cos \frac{\varphi}{2}$.

所求的弧长为

$$s = 2 \int_0^\pi 2a \cos \frac{\varphi}{2} d\varphi = 8a.$$

2449. $r = \frac{p}{1 + \cos\varphi} \left(|\varphi| \leq \frac{\pi}{2} \right)$.

解 $r' = \frac{p \sin\varphi}{(1 + \cos\varphi)^2}$,

$$\sqrt{r^2 + r'^2} = \frac{2p \cos \frac{\varphi}{2}}{(1 + \cos\varphi)^2}.$$

所求的弧长为

$$\begin{aligned}
s &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2p \cos \frac{\varphi}{2}}{(1 + \cos \varphi)^2} d\varphi = \frac{p}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^3 \frac{\varphi}{2} d\varphi \\
&= \frac{p}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec \frac{\varphi}{2} \left(1 + \operatorname{tg}^2 \frac{\varphi}{2} \right) d\varphi \\
&= p \left\{ \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos \frac{\varphi}{2}} \right. \\
&\quad \left. + 2 \int_0^{\frac{\pi}{2}} \sqrt{\sec^2 \frac{\varphi}{2} - 1} d\left(\sec \frac{\varphi}{2} \right) \right\} \\
&= 2p \left\{ \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{\varphi}{4} \right) + \frac{\sec \frac{\varphi}{2}}{2} \sqrt{\sec^2 \frac{\varphi}{2} - 1} \right. \\
&\quad \left. - \frac{1}{2} \ln \left(\sec \frac{\varphi}{2} + \operatorname{tg} \frac{\varphi}{2} \right) \right\} \Big|_0^{\frac{\pi}{2}} \\
&= p(\sqrt{2} + \ln(\sqrt{2} + 1)).
\end{aligned}$$

2450. $r = a \sin^3 \frac{\varphi}{3}.$

解 $\sqrt{r^2 + r'^2} = a \sin^2 \frac{\varphi}{3} (0 \leq \varphi \leq 3\pi)$ (图 4.37).

所求的弧长为

$$\begin{aligned}
s &= \int_0^{3\pi} a \sin^2 \frac{\varphi}{3} d\varphi \\
&= \frac{3\pi a}{2}.
\end{aligned}$$

我们甚至可以证明

1° 弧 \widehat{AB} 为弧 \widehat{OABC} 的三分之一;

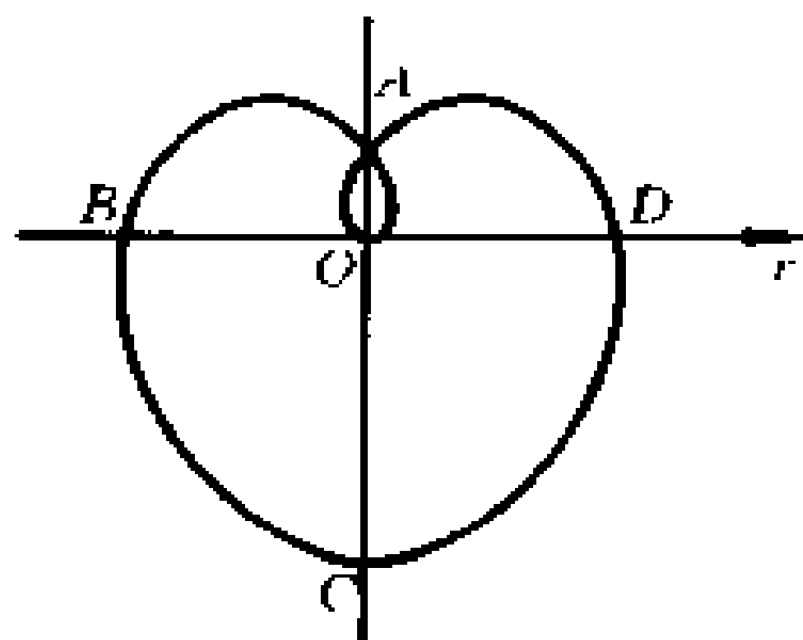


图 4.37

$2^\circ \widehat{OA}, \widehat{AB}, \widehat{BC}$ 之间依次是等差的, 其公差为 $\frac{3a}{8} \sqrt{3}$.

不仅如此, 我们还可证明更一般的情况:

曲线 $r = a \sin^n \left(\frac{\theta}{n} \right)$ (n 为正整数) 之全长为

$$s = \begin{cases} \frac{(2k-2)!!}{(2k-1)!!} 4ka, & \text{当 } n = 2k \text{ 时,} \\ \frac{(2k+1)!!}{(2k)!!} \pi a, & \text{当 } n = 2k+1 \text{ 时.} \end{cases}$$

2451. $r = a \operatorname{th} \frac{\varphi}{2} (0 \leq \varphi \leq 2\pi)$.

解 $r' = \frac{a}{2} \cdot \frac{1}{\operatorname{ch}^2 \frac{\varphi}{2}}.$

$$\begin{aligned} \sqrt{r^2 + r'^2} &= \frac{a}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \sqrt{4 \operatorname{sh}^2 \frac{\varphi}{2} \operatorname{ch}^2 \frac{\varphi}{2} + 1} \\ &= \frac{a}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \sqrt{\operatorname{sh}^2 \varphi + 1} \\ &= \frac{a \operatorname{ch} \varphi}{2 \operatorname{ch}^2 \frac{\varphi}{2}} = \frac{a \operatorname{ch} \varphi}{1 + \operatorname{ch} \varphi} \\ &= a \left(1 - \frac{1}{1 + \operatorname{ch} \varphi} \right) \\ &= a \left[1 - \frac{1}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \right]. \end{aligned}$$

所求的弧长为

$$s = \int_0^{2\pi} a \left[1 - \frac{1}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \right] d\varphi = a \left(\varphi - \operatorname{th} \frac{\varphi}{2} \right) \Big|_0^{2\pi}$$

$$= a \int_0^{2\pi} \sqrt{1 - \epsilon^2 \sin^2 t} dt.$$

对于正弦曲线, 其一波 (x 由 0 到 $2\pi b$) 之长为

$$\begin{aligned} s_2 &= \int_0^{2\pi b} \sqrt{1 + \frac{c^2}{b^2} \cos^2 \frac{x}{b}} dx \\ &= \int_0^{2\pi} \sqrt{b^2 + c^2 \cos^2 t} dt \\ &= \int_0^{2\pi} \sqrt{a^2 - c^2 \sin^2 t} dt \\ &= a \int_0^{2\pi} \sqrt{1 - \epsilon^2 \sin^2 t} dt. \end{aligned}$$

所以 $s_1 = s_2$, 本题得证.

2454. 抛物线 $4ay = x^2$ 沿 Ox 轴滚动, 证明抛物线的焦点划成悬链线.

解 如图 4.38 所示, 设抛物线切 Ox 轴于点 $A(s, 0)$, O' 为抛物线的顶点, P' 为焦点, $O'Y'$ 为对称轴, $O'X' \perp O'Y'$. 过 A 作 $AB \perp O'X'$.

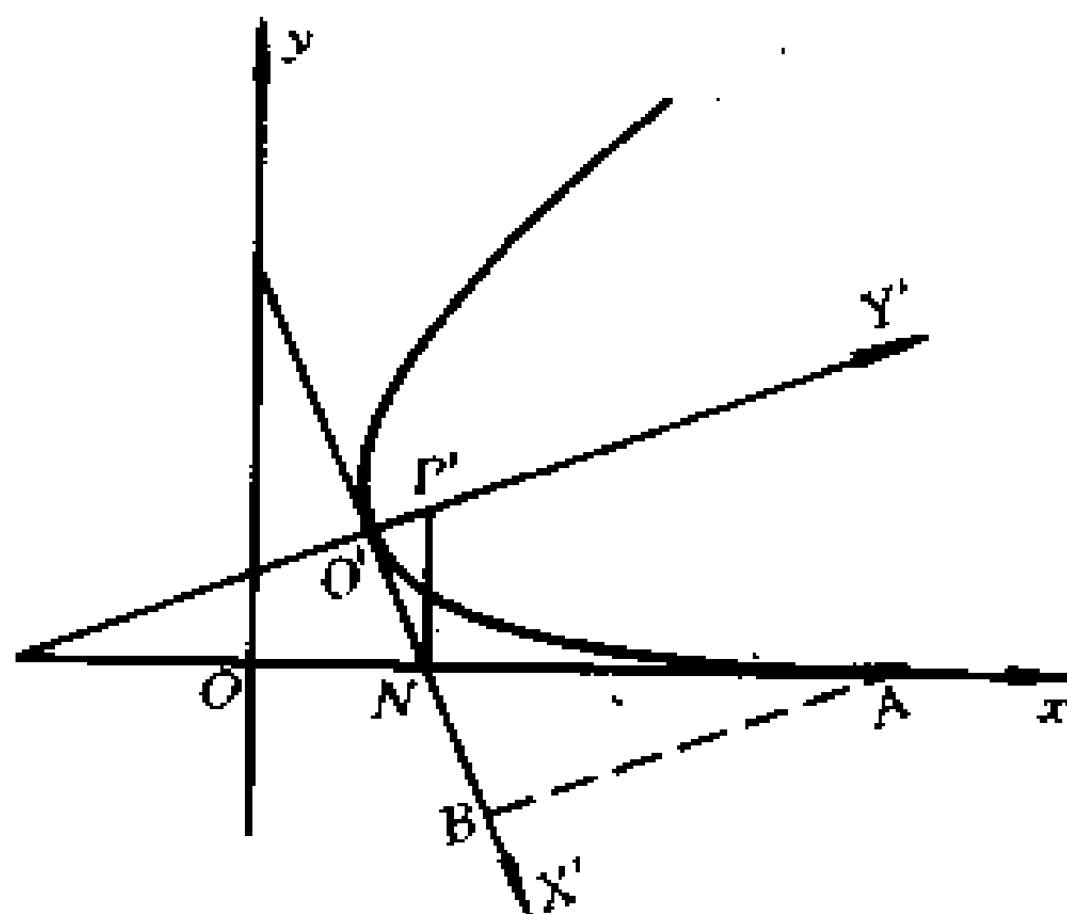


图 4.38

引入参数 $O'N = t$, 则由抛物线的性质易知:
 $P'N \perp Ox, O'B = 2O'N = 2t$. 从而有

$$AB = \frac{(2t)^2}{4a} = \frac{t^2}{a}, AN = t \cdot \sqrt{1 + \frac{t^2}{a^2}}$$

$$\begin{aligned} s &= \int_0^{2t} \sqrt{1 + \left(\frac{x}{2a}\right)^2} dx \\ &= t \sqrt{1 + \left(\frac{t}{a}\right)^2} + a \ln \left[\frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2} \right], \end{aligned}$$

$$P'N = a \sqrt{1 + \left(\frac{t}{a}\right)^2}.$$

于是, 焦点 P' 的坐标 x, y 由参数 t 表出:

$$\begin{cases} x = s - AN = a \ln \left[\frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2} \right], & (1) \end{cases}$$

$$\begin{cases} y = P'N = a \sqrt{1 + \left(\frac{t}{a}\right)^2}. & (2) \end{cases}$$

由(1)式得

$$e^{\frac{x}{a}} = \frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2},$$

$$e^{-\frac{x}{a}} = -\frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2}.$$

上面两式相加, 得

$$e^{\frac{x}{a}} + e^{-\frac{x}{a}} = 2 \sqrt{1 + \left(\frac{t}{a}\right)^2}.$$

再以(2)式代入上式, 最后得

$$y = \frac{a}{2} (e^{\frac{x}{a}} + e^{-\frac{x}{a}}) = a \operatorname{ch} \frac{x}{a}.$$

这说明抛物线的焦点划成悬链线.

2455. 求环线

$$y = \pm \left(\frac{1}{3} - x \right) \sqrt{x}$$

所包围的面积与周长等于这曲线的围线长的圆面积之比.

解 当 $x = 0$ 及 $x = \frac{1}{3}$ 时, $y = 0$. 此环线的面积为

$$S_1 = 2 \int_0^{\frac{1}{3}} \left(\frac{1}{3} - x \right) \sqrt{x} dx = \frac{8}{135 \sqrt{3}}.$$

此环线的周长为

$$\begin{aligned} s &= 2 \int_0^{\frac{1}{3}} \sqrt{1 + \left(\frac{1}{6\sqrt{x}} - \frac{3\sqrt{x}}{2} \right)^2} dx \\ &= 2 \int_0^{\frac{1}{3}} \left(\frac{1}{6\sqrt{x}} - \frac{3\sqrt{x}}{2} \right) dx \\ &= \frac{4}{3\sqrt{3}}. \end{aligned}$$

按题设有 $\frac{4}{3\sqrt{3}} = 2\pi R$, 所以 $R = \frac{2}{3\sqrt{3}\pi}$. 圆面积

$$S_2 = \pi R^2 = \frac{4}{27\pi}.$$

于是,

$$\frac{S_1}{S_2} = \frac{2\pi}{5\sqrt{3}} \doteq 0.73.$$

§ 7. 体积的计算法

1° 由已知横切面计算物体体积 若物体体积 V 存在及 $S = S(x)$

($a \leq x \leq b$) 为用平面切下的物体的横断面积, 而此横断面为经过 x 点垂直于 Ox 轴者, 则

$$V = \int_a^b S(x) dx.$$

2° 旋转体的体积 面积

$$a \leq x \leq b; 0 \leq y \leq y(x),$$

式中 $y(x)$ 为单值连续函数, 绕 Ox 轴旋转所成旋转体的体积等于

$$V_x = \pi \int_a^b y^2(x) dx.$$

更普遍的情形: 面积

$$a \leq x \leq b; y_1(x) \leq y \leq y_2(x),$$

式中 $y_1(x)$ 和 $y_2(x)$ 是非负连续函数, 绕 Ox 轴旋转所成的环形的体积等于

$$V = \pi \int_a^b [y_2^2(x) - y_1^2(x)] dx.$$

2456. 求顶楼的体积, 其底是边长等于 a 及 b 的矩形, 其顶的棱边等于 c , 而高等于 h .

解 如图 4.39 所示的顶楼, 取 x 轴向下, 则有

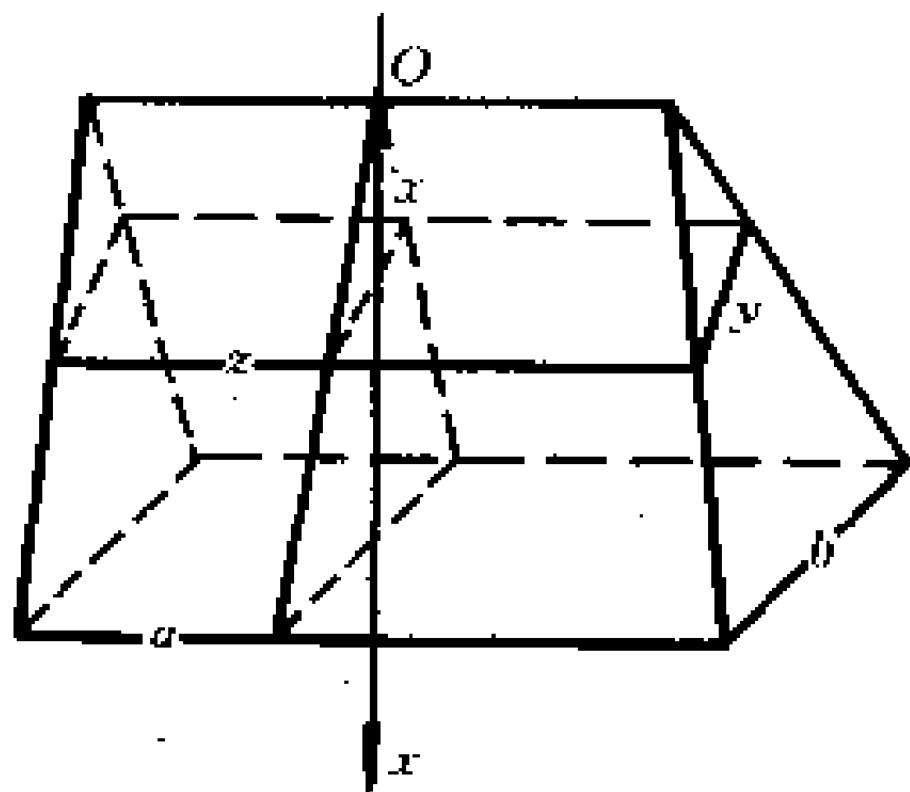


图 4.39

$$\frac{y}{b} = \frac{x}{h} \quad \text{或} \quad y = \frac{b}{h}x,$$

$$\frac{z-c}{a-c} = \frac{x}{h} \quad \text{或} \quad z = \frac{a-c}{h}x + c.$$

于是, 所求顶楼的体积为

$$\begin{aligned} V &= \int_0^h yz dx = \int_0^h \frac{b}{h}x \left(\frac{a-c}{h}x + c \right) dx \\ &= \frac{b}{h} \cdot \frac{a-c}{h} \cdot \frac{1}{3}h^3 + \frac{bc}{h} \cdot \frac{1}{2}h^2 \\ &= \frac{bh}{6}(2a+c). \end{aligned}$$

2457. 求截楔形的体积, 其平行的上下底为边长分别等于 A , B 和 a, b 的矩形, 而高等于 h .

解 如图 4.40 所示,

$$OO' = \frac{A}{2},$$

$$QQ' = \frac{a}{2},$$

$$OQ = h,$$

设 $OP = x$, 则

$$PP' = \frac{a}{2} + \frac{h-x}{h} \left(\frac{A-a}{2} \right).$$

同样可得

$$LP' = \frac{b}{2} + \frac{h-x}{h} \left(\frac{B-b}{2} \right).$$

从而

$$\text{面积 } KLMN = ab + (A-a)(B-b) \left(1 - \frac{x}{h} \right)^2$$

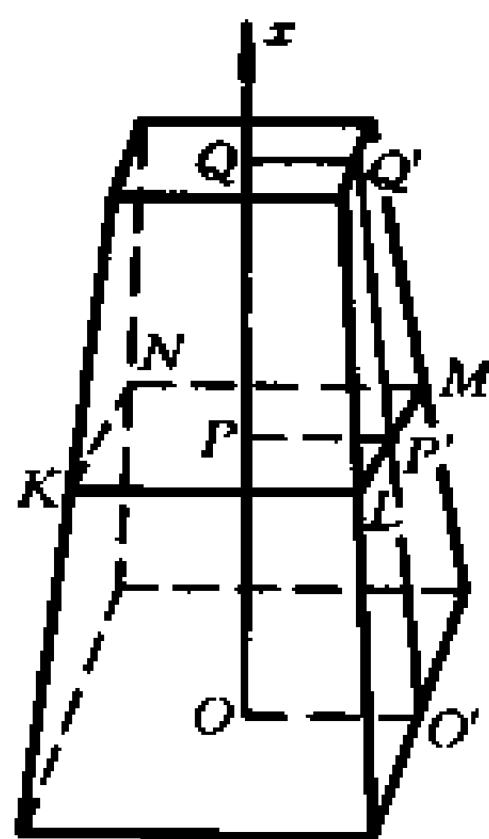


图 4.40

$$+ [a(B-b) + b(A-a)] \left(1 - \frac{x}{h}\right) = f(x).$$

所求截楔形的体积为

$$V = \int_0^h f(x) dx = \frac{h}{6} [(2A+a)B + (2a+A)b].$$

2458. 求截锥体的体积, 其上下底为半轴长分别等于 A, B 和 a, b 的椭圆, 而高等于 h .

解 同 2457 题, 任一平行于上下底且距离下底为 x 的截面为一椭圆, 其半轴分别为

$$a' = a + \left(1 - \frac{x}{h}\right)(A-a)$$

及

$$b' = b + \left(1 - \frac{x}{h}\right)(B-b),$$

从而此截面的面积为

$$\begin{aligned} S(x) &= \pi a' b' \\ &= \pi \left\{ ab + (A-a)(B-b) \left(1 - \frac{x}{h}\right)^2 \right. \\ &\quad \left. + [a(B-b) + b(A-a)] \left(1 - \frac{x}{h}\right) \right\}. \end{aligned}$$

所求的体积为

$$V = \int_0^h S(x) dx = \frac{\pi h}{6} [(2A+a)B + (A+2a)b].$$

2459. 求旋转抛物体的体积, 其底为 S , 而高等于 H .

解 不失一般性, 假设抛物线方程为

$$y^2 = 2px,$$

绕 Ox 轴旋转, 如图 4.41 所示.

记 $OA = H$,

$OB = x$, 按假设有

$$S = \pi \cdot \overline{AC}^2$$

$$= \pi(2pH)$$

$$= 2\pi p x,$$

距原点为 x 的截面面积为

$$S(x) = \pi y^2 = 2\pi p x.$$

于是, 所求的体积为

$$V = \int_0^H S(x) dx$$

$$= \pi p H^2 = \frac{SH}{2}.$$

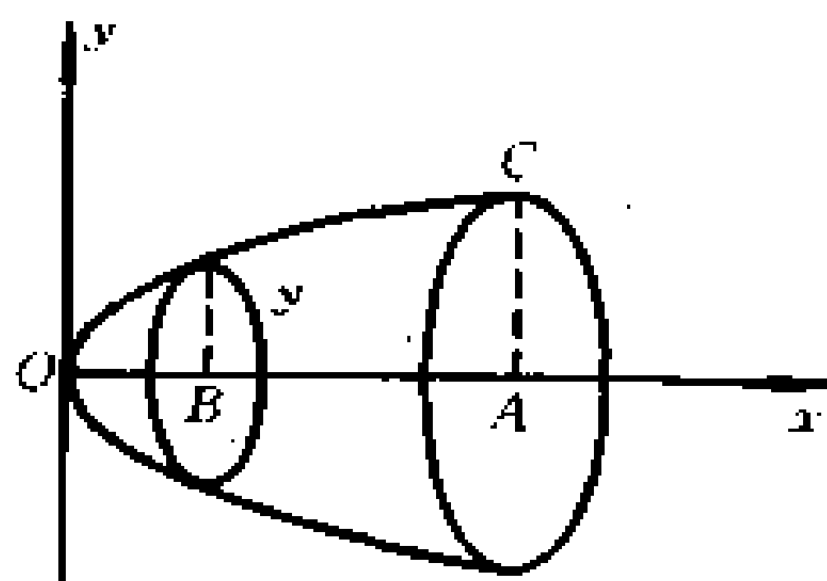


图 4.41

2460. 设立体之垂直于 Ox 轴的横截面的面积 $S \leftarrow S(x)$, 依下面的二次式规律变化:

$$S(x) = Ax^2 + Bx + C \quad (a \leq x \leq b),$$

其中 A, B 及 C 为常数.

证明此物体之体积等于

$$V = \frac{H}{6} \left[S(a) + 4S\left(\frac{a+b}{2}\right) + S(b) \right],$$

其中 $H = b - a$ (辛普森公式).

$$\begin{aligned} \text{证} \quad V &= \int_a^b (Ax^2 + Bx + C) dx \\ &= \frac{A}{3} (b^3 - a^3) + \frac{B}{2} (b^2 - a^2) + C(b - a) \\ &= \frac{b-a}{6} [2A(b^2 + ab + a^2) + 3B(a + b) + 6C] \\ &= \frac{H}{6} [(Aa^2 + Ba + C) + (Ab^2 + Bb + C) \\ &\quad + A(a^2 + 2ab + b^2) + 2B(a + b) + 4C] \end{aligned}$$

$$= \frac{H}{6} \left[S(a) + S(b) + 4S\left(\frac{a+b}{2}\right) \right].$$

2461. 物体是点 $M(x, y, z)$ 的集合, 其中 $0 \leq z \leq 1$, 而且若 z 为有理数时, $0 \leq x \leq 1, 0 \leq y \leq 1$; 若 z 为无理数时, $-1 \leq x \leq 0, -1 \leq y \leq 0$. 证明虽然对应的积分为

$$\int_0^1 S(z) dz = 1.$$

但此物体的体积不存在.

证 显然, 对任何 $0 \leq z \leq 1$, 不论 z 是有理数还是无理数, 都有 $S(z) = 1$. 从而

$$\int_0^1 S(z) dz = \int_0^1 dz = 1.$$

下证此物体 (V) 的体积不存在. 显然, 无完全含于 (V) 内的多面体 (X) 存在, 从而这种 (X) 的体积的上确界为零, 即 (V) 的内体积 $V_+ = \sup\{X\} = 0$. 另一方面, (V) 的外体积 $V^* = \inf\{Y\}$, 其中的下确界是对所有完全包含着 (V) 的多面体 (Y) 的体积 Y 来取的. 由于 $0 \leq z \leq 1$ 中的有理数和无理数都在 $0 \leq z \leq 1$ 中是稠密的, 故, 显然, 上述任何完全包含着 (V) 的多面体 (Y) 都必完全包含着点集 $(Y_0) = \{(x, y, z) | 0 \leq z \leq 1; 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ 以及 } -1 \leq x \leq 0, -1 \leq y \leq 0\}$. 而 (Y_0) 又完全包含着 (V) , 并且 (Y_0) 的体积 $Y_0 = 2$. 由此可知 $V^* = \inf\{Y\} = 2$. 于是 $V_+ \neq V^*$. 故 (V) 的体积不存在.

求下列曲面所围成的体积:

2462. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = \frac{c}{a}x, z = 0.$

解 如图 4.42 所示, 用垂直 Oy 轴的平面截割, 得一直角三角形 PQR .

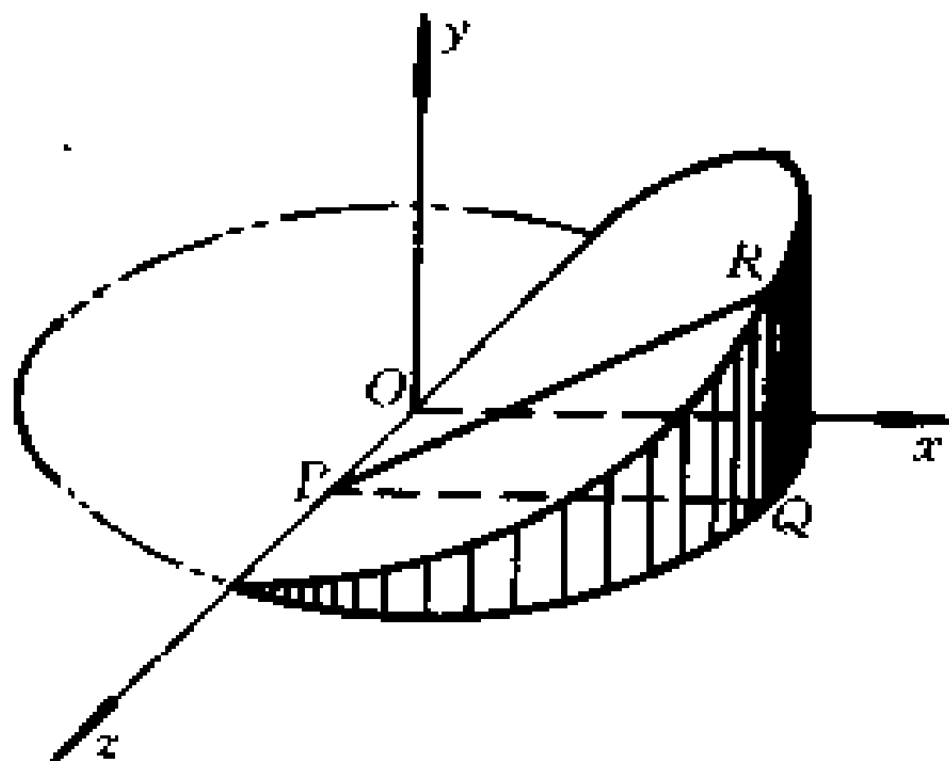


图 4.42

设 $OP = y$, 则高 $QR = \frac{c}{a}x$, 从而它的面积为

$$\frac{1}{2} \cdot \frac{c}{a}x^2 = \frac{ac}{2} \left(1 - \frac{y^2}{b^2} \right).$$

于是, 所求体积为

$$V = 2 \int_0^b \frac{ac}{2} \left(1 - \frac{y^2}{b^2} \right) dy = \frac{2}{3}abc.$$

2463. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (椭球).

解 用垂直于 Ox 轴的平面截椭球得截痕为一椭圆, 它在 yoz 平面上的投影为

$$\frac{y^2}{b^2 \left(1 - \frac{x^2}{a^2} \right)} + \frac{z^2}{c^2 \left(1 - \frac{x^2}{a^2} \right)} = 1.$$

由此显见其半轴分别为

$$V = 8 \int_0^a (a^2 - z^2) dz = \frac{16}{3} a^3.$$

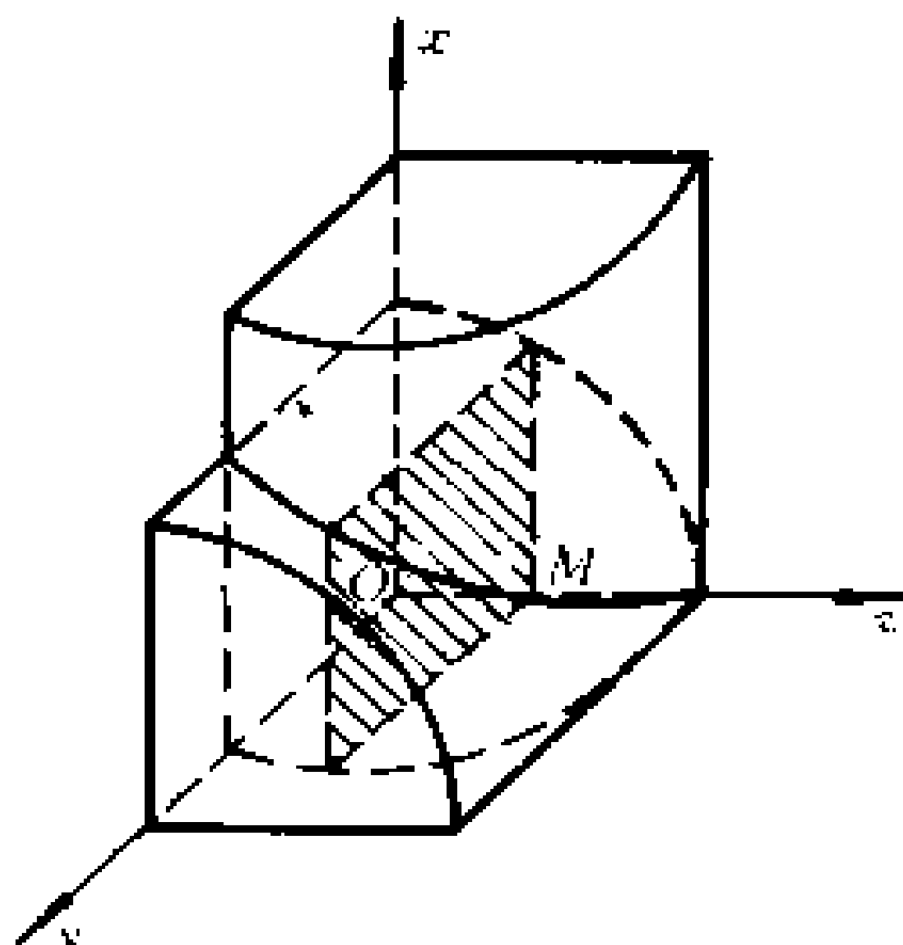


图 4.43

2466. $x^2 + y^2 + z^2 = a^2, x^2 + y^2 = ax.$

解 如图 4.44 所示, 过点 $M(x, 0, 0)$ 垂直于 Ox 轴作一平面, 在所给立体上截出一曲边梯形, 其曲边由方程

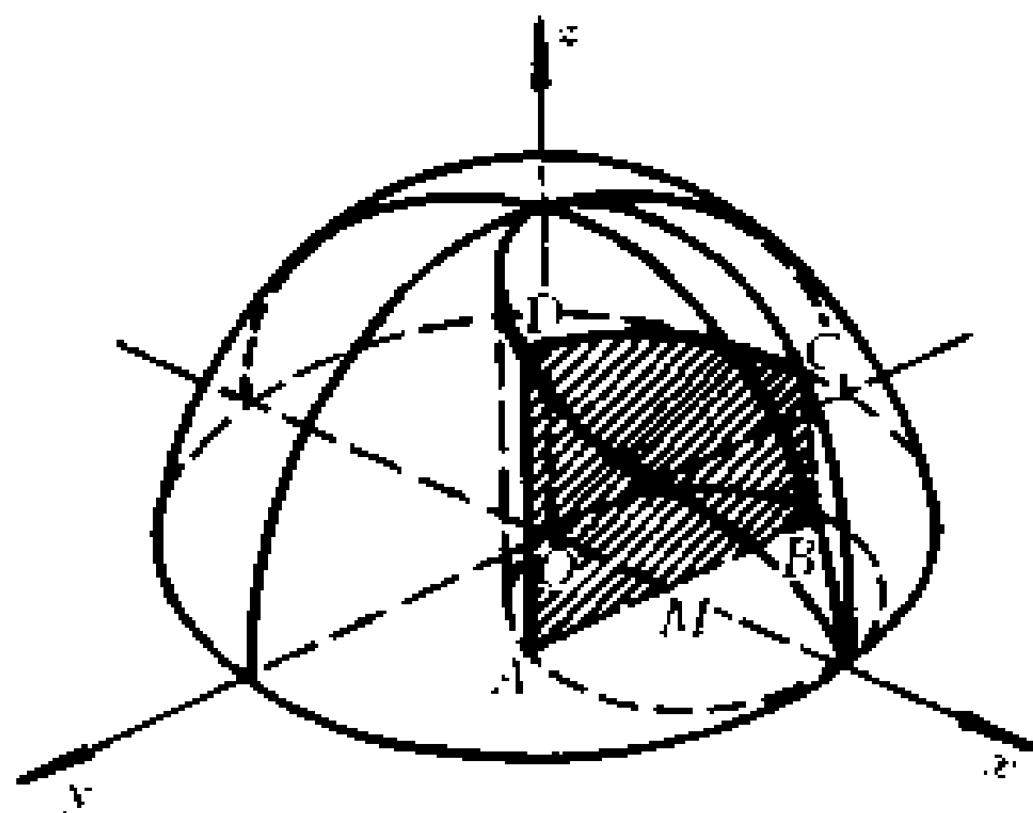


图 4.44

$$z = \sqrt{(a^2 - x^2) - y^2}$$

给出(上半面),

其变化范围为:

$$-\sqrt{ax - x^2} \leq y \leq \sqrt{ax - x^2} \text{ (如图中 } ABCD \text{)}.$$

从而其截面积为

$$\begin{aligned} S(x) &= 2 \int_0^{\sqrt{ax-x^2}} \sqrt{(a^2 - x^2) - y^2} dy \\ &= a^{\frac{3}{2}} x^{\frac{1}{2}} - a^{\frac{1}{2}} x^{\frac{3}{2}} + (a^2 - x^2) \arcsin \sqrt{\frac{x}{a+x}}. \end{aligned}$$

于是,所求的体积为

$$\begin{aligned} V &= 2 \int_0^a S(x) dx \\ &= 2 \int_0^a \left[a^{\frac{3}{2}} x^{\frac{1}{2}} - a^{\frac{1}{2}} x^{\frac{3}{2}} \right. \\ &\quad \left. + (a^2 - x^2) \arcsin \sqrt{\frac{x}{a+x}} \right] dx \\ &= 4 \left\{ \frac{1}{3} a^3 - \frac{1}{5} a^3 + \left[\left(\frac{\pi a^3}{4} - \frac{1}{2} a^3 \right) \right. \right. \\ &\quad \left. \left. - \left(\frac{1}{12} \pi a^3 - \frac{13}{90} a^3 \right) \right] \right\} \\ &= \frac{2}{3} a^3 \left(\pi - \frac{4}{3} \right). \end{aligned}$$

2467. $z^2 = b(a - x), x^2 + y^2 = ax.$

解 先求体积的四分之一部分,截面积为

$$\begin{aligned} S(x) &= \int_0^{\sqrt{ax-x^2}} \sqrt{b(a-x)} dy \\ &= \sqrt{ax - x^2} \cdot \sqrt{b(a-x)}. \end{aligned}$$

从而

$$\begin{aligned}\frac{1}{4}V &= \int_0^a S(x)dx = \int_0^a \sqrt{ax-x^2} \cdot \sqrt{b(a-x)}dx \\ &= \sqrt{b} \int_0^a \sqrt{x}(a-x)dx \\ &= \frac{4}{15}a^2 \sqrt{ab}.\end{aligned}$$

于是,所求的体积为

$$V = \frac{16}{15}a^2 \sqrt{ab}.$$

2468. $\frac{x^2}{a^2} + \frac{y^2}{z^2} = 1 (0 < z < a).$

解 固定 z , 则截面为一椭圆, 其面积为

$$P(z) = \pi az.$$

于是, 所求的体积为

$$V = \int_0^a P(z)dz = \pi a \int_0^a z dz = \frac{\pi a^3}{2}.$$

2469⁺. $x + y + z^2 = 1, x = 0, y = 0, z = 0.$

解 固定 z , 则截面为一直角三角形, 其面积为

$$P(z) = \frac{1}{2}(1 - z^2)^2.$$

故所求体积

$$\begin{aligned}V &= \int_0^1 \frac{1}{2}(1 - z^2)^2 dz \\ &= \frac{1}{2} \int_0^1 (1 - 2z^2 + z^4) dz = \frac{4}{15}.\end{aligned}$$

注意, 曲面 $x + y + z^2 = 1$ 关于平面 $z = 0$ 对称, 故它与三个平面 $x = 0, y = 0, z = 0$ 围成的图形有两个, 一个位于 Oxy 平面之上, 一个位于 Oxy 平面之

下,彼此是对称的(关于 Oxy 平面),从而它们的体积相等. 我们以上求的是位于 Oxy 平面之上的那一个图形的体积.

$$2470. \quad x^2 + y^2 + z^2 + xy + yz + zx = a^2.$$

解 不妨设 $a > 0$. 此为一有心椭球面. 固定 z , 得在平面 xoy 上的投影为

$$x^2 + xy + y^2 + zx + zy + (z^2 - a^2) = 0,$$

此截面的面积为

$$S(z) = - \frac{\pi \Delta}{\left(1 - \frac{1}{4}\right)^{\frac{3}{2}}} = - \frac{8\pi \Delta}{3\sqrt{3}},$$

其中

$$\Delta = \begin{vmatrix} 1 & \frac{1}{2} & \frac{z}{2} \\ \frac{1}{2} & 1 & \frac{z}{2} \\ \frac{z}{2} & \frac{z}{2} & z^2 - a^2 \end{vmatrix} = \frac{2z^2 - 3a^2}{4},$$

所以

$$S(z) = \frac{2(3a^2 - 2z^2)\pi}{3\sqrt{3}}.$$

z 的变化范围为适合下述不等式的集合:

$$2z^2 - 3a^2 \leq 0,$$

即

$$|z| \leq \sqrt{\frac{3}{2}}a.$$

于是,所求的体积为

$$V = \int_{-\sqrt{\frac{3}{2}|a|}}^{\sqrt{\frac{3}{2}|a|}} \frac{2(3a^2 - 2z^2)\pi}{3\sqrt{3}} dz = \frac{4\sqrt{2}\pi}{3} a^3.$$

*) 此公式详见 Г. М 菲赫金哥尔茨著《微积分学教程》第二卷第一分册第 330 目 7.

2471. 证明: 将面积

$$a \leq x \leq b, 0 \leq y \leq y(x),$$

(式中 $y(x)$ 为连续函数) 绕 Oy 轴旋转所成的旋转体体积等于

$$V_y = 2\pi \int_a^b xy(x) dx.$$

$$\begin{aligned} \text{证 } \Delta V_y &= \pi[(x + \Delta x)^2 - x^2]y(x) \\ &= 2\pi xy(x)\Delta x. \end{aligned}$$

于是, 所求的体积为

$$V_y = 2\pi \int_a^b xy(x) dx.$$

求下列曲线旋转所成曲面包围的体积:

2472. $y = b\left(\frac{x}{a}\right)^{\frac{2}{3}} (0 \leq x \leq a)$ 绕 Ox 轴(半三次抛物线).

解 所求的体积为

$$V_x = \pi b^2 \int_0^a \left(\frac{x}{a}\right)^{\frac{4}{3}} dx = \frac{3}{7} \pi ab^2.$$

2473. $y = 2x - x^2, y = 0$; (a) 绕 Ox 轴; (6) 绕 Oy 轴.

解 令 $y = 0$ 得 $x = 0$ 或 $x = 2$.

于是, 所求的体积为

$$(a) V_x = \pi \int_0^2 (2x - x^2)^2 dx = \frac{16\pi}{15};$$

$$(6) V_y = 2\pi \int_0^2 x(2x - x^2) dx = \frac{8\pi}{3}.$$

2474. $y = \sin x, y = 0 (0 \leq x \leq \pi)$; (a) 绕 Ox 轴; (6) 绕 Oy 轴.

解 所求的体积为

$$(a) V_x = \pi \int_0^{\pi} \sin^2 x dx = \frac{\pi^2}{2};$$

$$(6) V_y = 2\pi \int_0^{\pi} x \sin x dx = 2\pi^2.$$

2475. $y = b \left(\frac{x}{a} \right)^2, y = b \left| \frac{x}{a} \right|$; (a) 绕 Ox 轴; (6) 绕 Oy 轴.

解 交点为 (a, b) 及 $(-a, b)$.

所求的体积为

$$(a) V_x = 2\pi \int_0^a \left(b^2 \frac{x^2}{a^2} - b^2 \frac{x^4}{a^4} \right) dx \\ = \frac{4\pi}{15} ab^2;$$

$$(6) V_y = \pi \int_0^b \left(\frac{a^2 y}{b} - \frac{a^2 y^2}{b^2} \right) dy \\ = \frac{\pi a^2 b}{6}.$$

2476. $y = e^{-x}, y = 0 (0 \leq x < +\infty)$; (a) 绕 Ox 轴; (6) 绕 Oy 轴.

解 所求的体积为

$$(a) V_x = \pi \int_0^{+\infty} e^{-2x} dx = \frac{\pi}{2};$$

$$(6) V_y = \pi \int_0^1 (-\ln y)^2 dy = 2\pi.$$

2477. $x^2 + (y - b)^2 = a^2 (0 < a \leq b)$ 绕 Ox 轴.

解 $y_1 = b + \sqrt{a^2 - x^2}, y_2 = b - \sqrt{a^2 - x^2}$
 $(-a \leq x \leq a).$

所求的体积为

$$\begin{aligned} V_x &= \pi \int_{-a}^a (y_1^2 - y_2^2) dx \\ &= 8b\pi \int_0^a \sqrt{a^2 - x^2} dx = 2\pi^2 a^2 b. \end{aligned}$$

2478. $x^2 - xy + y^2 = a^2$ 绕 Ox 轴.

解 原方程即 $y^2 - xy + x^2 - a^2 = 0$, 从而

$$y = \frac{x \pm \sqrt{4a^2 - 3x^2}}{2},$$

函数的定义域为 $\left[-\frac{2}{\sqrt{3}}a, \frac{2}{\sqrt{3}}a\right]$. 与 Ox 轴的交点分别为 $x = -a$ 与 $x = a$.

于是, 所求的体积为

$$\begin{aligned} V_x &= 2 \left\{ \pi \int_0^a \frac{1}{4} (x + \sqrt{4a^2 - 3x^2})^2 dx \right. \\ &\quad \left. + \pi \int_a^{\frac{2}{\sqrt{3}}a} \left[\frac{1}{4} (x + \sqrt{4a^2 - 3x^2})^2 \right. \right. \\ &\quad \left. \left. - \frac{1}{4} (x - \sqrt{4a^2 - 3x^2})^2 \right] dx \right\} \\ &= \frac{\pi}{2} \int_0^a (4a^2 - 2x^2 + 2x \sqrt{4a^2 - 3x^2}) dx \\ &\quad + 2\pi \int_a^{\frac{2}{\sqrt{3}}a} x \sqrt{4a^2 - 3x^2} dx \\ &= \pi \left(2a^3 - \frac{1}{3}a^3 - \frac{1}{9} (4a^2 - 3x^2)^{\frac{3}{2}} \right) \Big|_0^a \\ &\quad - \frac{2}{9} (4a^2 - 3x^2)^{\frac{3}{2}} \Big|_a^{\frac{2}{\sqrt{3}}a} = \frac{8}{3} \pi a^3. \end{aligned}$$

2479. $y = e^{-x} \sqrt{\sin x} (0 \leq x < +\infty)$ 绕 Ox 轴.

解 函数定义域为 $[2n\pi, (2n+1)\pi]$, ($n = 0, 1, 2,$

...). 故所求的体积为

$$\begin{aligned}
 V_x &= \pi \sum_{n=0}^{\infty} \int_{2n\pi}^{(2n+1)\pi} e^{-2x} \sin x dx \\
 &= \sum_{n=0}^{\infty} \frac{\pi}{5} e^{-2x} (-2\sin x - \cos x) \Big|_{2n\pi}^{(2n+1)\pi} \\
 &= \frac{\pi}{5} (e^{-2\pi} + 1) \sum_{n=0}^{\infty} e^{-4n\pi} \\
 &= \frac{\pi}{5} \cdot \frac{e^{-2\pi} + 1}{1 - e^{-4\pi}} = \frac{\pi}{5(1 - e^{-2\pi})}.
 \end{aligned}$$

2480. $x = a(t - \sin t), y = a(1 - \cos t) (0 \leq t \leq 2\pi),$
 $y = 0;$

(a) 绕 Ox 轴; (б) 绕 Oy 轴; (в) 绕直线 $y = 2a.$

解 所求的体积为

$$(a) \quad V_x = \pi \int_0^{2\pi} a^3 (1 - \cos t)^3 dt = 5\pi^2 a^3;$$

$$\begin{aligned}
 (б) \quad V_y &= 2\pi \int_0^{2\pi} a^3 (t - \sin t) (1 - \cos t)^2 dt \\
 &= 6\pi^3 a^3;
 \end{aligned}$$

(в) 作平移: $y = \bar{y} + 2a, x = \bar{x},$ 则曲线方程为
 $\bar{x} = a(t - \sin t), \bar{y} = -a(1 + \cos t),$ 及
 $\bar{y} = -2a.$

于是, 所求的体积为

$$\begin{aligned}
 V_{\bar{y}} &= \pi \int_0^{2\pi} [4a^2 - a^2(1 + \cos t)^2] a(1 - \cos t) dt \\
 &= 7\pi^2 a^3.
 \end{aligned}$$

2481. $x = a \sin^3 t, y = b \cos^3 t (0 \leq t \leq 2\pi);$

(a) 绕 Ox 轴; (б) 绕 Oy 轴.

解 所求的体积为

$$\begin{aligned}
(a) \quad V_x &= 2\pi \int_0^{\frac{\pi}{2}} (b^2 \cos^6 t) (3a \sin^2 t \cos t) dt \\
&= 6\pi ab^2 \left(\int_0^{\frac{\pi}{2}} \cos^7 t dt - \int_0^{\frac{\pi}{2}} \cos^9 t dt \right) \\
&= 6\pi ab^2 \left(\frac{6!!}{7!!} - \frac{8!!}{9!!} \right)^{*)} = \frac{32}{105} \pi ab^2;
\end{aligned}$$

(6) 利用对称性, 只须将上述答案中 a, b 对调即得

$$V_y = \frac{32}{105} \pi a^2 b.$$

*) 利用 2282 题的结果.

2482. 证明把面积

$$0 \leq \alpha \leq \varphi \leq \beta \leq \pi, \quad 0 \leq r \leq r(\varphi)$$

(φ 与 r 为极坐标) 绕极轴旋转所成的体积等于

$$V = \frac{2\pi}{3} \int_{\alpha}^{\beta} r^3(\varphi) \sin \varphi d\varphi.$$

证 证法一:

微面积 $dS = r d\varphi dr$ 绕极轴旋转所得微环形体积

$$dV = 2\pi r \sin \varphi dS = 2\pi r^2 \sin \varphi d\varphi dr.$$

于是, 所求的体积

$$\begin{aligned}
V &= 2\pi \int_{\alpha}^{\beta} \sin \varphi d\varphi \int_{\alpha}^{r(\varphi)} r^2 dr \\
&= \frac{2\pi}{3} \int_{\alpha}^{\beta} r^3(\varphi) \sin \varphi d\varphi.
\end{aligned}$$

证法二:

应用直角坐标系下的古尔金第二定理^{*)}来证明.
对于微小面积元, 它的重心可以看成在点 $\left(\frac{2}{3} r \cos \varphi, \frac{2}{3} r \sin \varphi \right)$ 处(图 4.45).

$$\begin{aligned}
& + \frac{4\pi a^3}{3} \int_0^\pi \cos^4 \varphi d\varphi + \frac{\pi^2 a^3}{2} \\
& = \left(4\pi a^3 + \frac{\pi a^3}{2} \right) \frac{\pi}{2} + \frac{4\pi a^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \pi + \frac{\pi^2 a^3}{2} \\
& = \frac{13}{4} \pi^2 a^3.
\end{aligned}$$

注 (1) 在 V 的表达式中 $\frac{2}{3} r \cos \varphi$ 的系数 $\frac{2}{3}$ 是把微小面积集中在其重心 $\left(\frac{2}{3} r, \varphi \right)$ 处得出的.

$$(2) \int_0^\pi \cos^{2k+1} \varphi d\varphi = 0,$$

$$\int_0^\pi \cos^{2k} \varphi d\varphi = \frac{(2k-1)(2k-3)\cdots 3 \cdot 1}{2k(2k-2)\cdots 4 \cdot 2} \pi.$$

方法二:

心脏线 $r = a(1 + \cos \varphi)$ 的面积为 $\frac{3\pi a^2}{2}$ ^{*}, 而其重

心为 $\varphi_0 = 0, r_0 = \frac{5}{6}a$ ^{**}. 根据古尔金第二定理可得所求的体积为

$$V = 2\pi \left(\frac{5a}{6} + \frac{a}{4} \right) \frac{3\pi a^2}{2} = \frac{13}{4} \pi^2 a^3.$$

*) 利用 2419 题的结果.

**) 利用 2512 题的结果.

2484. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$;

(a) 绕 Ox 轴; (б) 绕 Oy 轴; (в) 绕直线 $y = x$.

解 (a) 曲线的极坐标方程为

$$r^2 = a^2(2\cos^2 \varphi - 1).$$

$$V_x = 2 \cdot \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} [a^2(2\cos^2 \varphi - 1)]^{\frac{3}{2}} \sin \varphi d\varphi.$$

由于

$$\begin{aligned}
 & \int (2\cos^2\varphi - 1)^{\frac{3}{2}} \sin\varphi d\varphi \\
 &= \int [(\sqrt{2} \cos\varphi)^2 - 1]^{\frac{3}{2}} d(\sqrt{2} \cos\varphi) \cdot \left(-\frac{1}{\sqrt{2}} \right) \\
 &= -\frac{1}{\sqrt{2}} \left[\frac{\sqrt{2} \cos\varphi}{8} (4\cos^2\varphi - 5) \sqrt{2\cos^2\varphi - 1} \right. \\
 &\quad \left. + \frac{3}{8} \ln(\sqrt{2} \cos\varphi + \sqrt{2\cos^2\varphi - 1}) \right] + C.
 \end{aligned}$$

所以

$$\begin{aligned}
 V_x &= \frac{4\pi a^3}{3\sqrt{2}} \left[\frac{\sqrt{2} \cos\varphi}{8} (4\cos^2\varphi - 5) \sqrt{2\cos^2\varphi - 1} \right. \\
 &\quad \left. + \frac{3}{8} \ln(\sqrt{2} \cos\varphi + \sqrt{2\cos^2\varphi - 1}) \right] \bigg|_0^{\frac{\pi}{4}} \\
 &= \frac{4\pi a^3}{3\sqrt{2}} \left(\frac{3}{8} \ln(\sqrt{2} + 1) - \frac{\sqrt{2}}{8} \right) \\
 &= \frac{1}{4} \pi a^3 \left[\sqrt{2} \ln(\sqrt{2} + 1) - \frac{2}{3} \right].
 \end{aligned}$$

(6) 利用对称性知,所求的体积为

$$\begin{aligned}
 V_y &= \frac{4\pi}{3} \int_0^{\frac{\pi}{4}} r^3 \cos\varphi d\varphi \\
 &= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} \sqrt{\cos^3 2\varphi} \cos\varphi d\varphi.
 \end{aligned}$$

令 $\sin\varphi = \frac{1}{\sqrt{2}} \sin x$, 则 $\sqrt{\cos 2\varphi} = \cos x$,

$$\cos\varphi d\varphi = \frac{1}{\sqrt{2}}\cos x dx, \text{ 并且 } x \text{ 的变化范围为 } \left(0, \frac{\pi}{2}\right).$$

于是,得

$$\begin{aligned} V &= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}} \cos^4 x dx \\ &= \frac{4\pi a^3}{3} \cdot \frac{1}{\sqrt{2}} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} \\ &= \frac{\pi^2 a^3}{4\sqrt{2}}. \end{aligned}$$

(B) 利用对称性知所求的体积为

$$\begin{aligned} V &= \frac{4\pi}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3 \sin\left(\frac{\pi}{4} - \varphi\right) d\varphi \\ &= \frac{4\pi a^3}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos^3 2\varphi} \left(\frac{1}{\sqrt{2}} \cos\varphi \right. \\ &\quad \left. - \frac{1}{\sqrt{2}} \sin\varphi \right) d\varphi \\ &= \frac{4\pi a^3}{3\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos^3 2\varphi} \cos\varphi d\varphi. \end{aligned}$$

若用本题(6)的变换,即得

$$\begin{aligned} V &= \frac{4\pi a^3}{3\sqrt{2}} 2 \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}} \cos^4 x dx \\ &= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}} \cos^4 x dx \\ &= \frac{4\pi a^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4}. \end{aligned}$$

2485. 求绕极轴把面积

$$a \leq r \leq a\sqrt{2\sin 2\varphi}$$

旋转而成的旋转体体积.

解 $r = a$ 与 $r = a \sqrt{2\sin 2\varphi}$, 在第一象限部分的交点的极角分别为 $\alpha = \frac{\pi}{12}$ 及 $\beta = \frac{5\pi}{12}$. 利用对称性知, 所求的体积应为

$$\begin{aligned} V &= \frac{4\pi}{3} \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} [(a \sqrt{2\sin 2\varphi})^3 - a^3] \sin \varphi d\varphi \\ &= \frac{4\pi a^3}{3} \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (4 \sqrt{2} \sqrt{\sin 2\varphi} \sin^2 \varphi \cos \varphi \\ &\quad - \sin \varphi) d\varphi. \end{aligned}$$

为求上述积分, 令

$$I_1 = \int \sqrt{\sin 2\varphi} \sin^2 \varphi \cos \varphi d\varphi,$$

$$I_2 = \int \sqrt{\sin 2\varphi} \cos^2 \varphi \cos \varphi d\varphi,$$

$$\text{则 } I_2 - I_1 = \frac{1}{3} \cos \varphi (\sin 2\varphi)^{\frac{3}{2}} + \frac{2}{3} I_1$$

即

$$I_2 - \frac{5}{3} I_1 = \frac{1}{3} \cos \varphi \cdot (\sin 2\varphi)^{\frac{3}{2}}. \quad (1)$$

又

$$\begin{aligned} I_2 + I_1 &= \int \sqrt{\sin 2\varphi} \cos \varphi d\varphi \\ &= \sqrt{2} \int \frac{\operatorname{tg} \varphi}{1 + \operatorname{tg}^2 \varphi} \sqrt{\operatorname{ctg} \varphi} d\varphi. \end{aligned}$$

令 $\operatorname{tg} \varphi = t$, 就可将上述积分化成二项式的微分的积分. 积分之, 得

$$I_2 + I_1 = \frac{1}{2} \sin \varphi \sqrt{\sin 2\varphi} + \frac{1}{2} \ln(\sin \varphi + \cos \varphi)$$

$$\begin{aligned}
& - \sqrt{\sin 2\varphi}) + \frac{1}{4} [\ln(\sin\varphi + \cos\varphi \\
& + \sqrt{\sin 2\varphi}) + \arcsin(\sin\varphi \\
& - \cos\varphi)].
\end{aligned} \tag{2}$$

(2) - (1), 得

$$\begin{aligned}
I_1 = \frac{3}{8} \left\{ \frac{1}{2} \sin\varphi \sqrt{\sin 2\varphi} + \frac{1}{2} \ln(\sin\varphi + \cos\varphi \right. \\
& - \sqrt{\sin 2\varphi} + \frac{1}{4} [\ln(\sin\varphi + \cos\varphi \\
& + \sqrt{\sin 2\varphi}) + \arcsin(\sin\varphi - \cos\varphi)] \\
& \left. - \frac{1}{3} \cos\varphi \cdot (\sin 2\varphi)^{\frac{3}{2}} \right\} + C.
\end{aligned}$$

从而, 得

$$\begin{aligned}
& \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \sqrt{\sin 2\varphi} \sin^2 \varphi \cos \varphi d\varphi \\
& = \frac{1}{8} + \frac{3}{64} \pi.
\end{aligned}$$

因此, 所求的体积为

$$\begin{aligned}
V &= \frac{4\pi a^3}{3} \left[4 \sqrt{2} \left(\frac{1}{8} + \frac{3\pi}{64} \right) + \cos\varphi \right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \\
&= \frac{\pi^2 a^3}{2 \sqrt{2}}.
\end{aligned}$$

§ 8. 旋转曲面表面积的计算法

平滑的曲线 AB 绕 Ox 轴旋转所成曲面的面积等于

$$P = 2\pi \int_A^B y ds,$$

式中 ds 为弧的微分.

求旋转下列曲线所成曲面的面积:

2486. $y = x \sqrt{\frac{x}{a}} (0 \leq x \leq a)$ 绕 Ox 轴.

解 $\sqrt{1+y'^2} = \sqrt{1 + \frac{9x}{4a}}$.

于是, 所求的表面积为

$$\begin{aligned} P_s &= 2\pi \int_0^a x \sqrt{\frac{x}{a}} \sqrt{1 + \frac{9x}{4a}} dx \\ &= \frac{3\pi}{a} \int_0^a x \sqrt{x^2 + \frac{4ax}{9}} dx \\ &= \frac{3\pi}{a} \int_0^a \left(x + \frac{2a}{9} \right) \sqrt{\left(x + \frac{2a}{9} \right)^2 - \left(\frac{2a}{9} \right)^2} d\left(x + \frac{2a}{9} \right) - \frac{3\pi}{a} \cdot \frac{2a}{9} \int_0^a \sqrt{x^2 + \frac{4ax}{9}} dx \\ &= \frac{3\pi}{a} \left[\frac{1}{3} \left(x^2 + \frac{4ax}{9} \right)^{\frac{3}{2}} \right]_0^a \\ &\quad - \frac{2\pi}{3} \left\{ \frac{x + \frac{2a}{9}}{2} \sqrt{x^2 + \frac{4ax}{9}} \right. \\ &\quad \left. - \frac{\frac{4a^2}{81}}{2} \ln \left[x + \frac{2a}{9} + \sqrt{x^2 + \frac{4ax}{9}} \right] \right\} \Big|_0^a \\ &= \frac{13}{27} \frac{\sqrt{13}}{\pi a^2} - \frac{11}{81} \frac{\sqrt{13}}{\pi a^2} + \frac{4\pi a^2}{243} \ln \frac{11 + 3\sqrt{13}}{2} \\ &= \frac{4\pi a^2}{243} \left(21\sqrt{13} + 2 \ln \frac{3 + \sqrt{13}}{2} \right). \end{aligned}$$

2487. $y = a \cos \frac{\pi x}{2b} (|x| \leq b)$ 绕 Ox 轴.

解 $y' = -\frac{\pi a}{2b} \sin \frac{\pi x}{2b},$

$$\sqrt{1+y'^2} = \frac{1}{2b} \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}}.$$

于是,所求的表面积为

$$\begin{aligned} P_x &= 2\pi \int_{-b}^b y \sqrt{1+y'^2} dx \\ &= 2\pi \int_{-b}^b \frac{a}{2b} \cos \frac{\pi x}{2b} \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}} dx \\ &= \frac{4}{\pi} \left[\frac{1}{2} \pi a \sin \frac{\pi x}{2b} \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}} \right. \\ &\quad \left. + \frac{4b^2}{2} \ln \left| \pi a \sin \frac{\pi x}{2b} + \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}} \right| \right] \Big|_0^b \\ &= 2a \sqrt{\pi^2 a^2 + 4b^2} + \frac{8b^2}{\pi} \ln \frac{\pi a + \sqrt{4b^2 + \pi^2 a^2}}{2b}. \end{aligned}$$

2488. $y = \operatorname{tg} x \left(0 \leq x \leq \frac{\pi}{4} \right)$ 绕 Ox 轴.

解 $\sqrt{1+y'^2} = \sqrt{1+\sec^4 x} = \frac{\sqrt{\cos^4 x + 1}}{\cos^2 x}.$

于是,所求的表面积为

$$\begin{aligned} P_x &= 2\pi \int_0^{\frac{\pi}{4}} \operatorname{tg} x \cdot \frac{\sqrt{\cos^4 x + 1}}{\cos^2 x} dx \\ &= \pi \int_0^{\frac{\pi}{4}} \sqrt{\cos^4 x + 1} d\left(\frac{1}{\cos^2 x}\right) \\ &= \pi \left[\frac{\sqrt{\cos^4 x + 1}}{\cos^2 x} - \ln(\cos^2 x + \sqrt{\cos^4 x + 1}) \right] \Big|_0^{\frac{\pi}{4}} \\ &= \pi \left[\sqrt{5} - \sqrt{2} + \ln \frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2} \right]. \end{aligned}$$

2489. $y^2 = 2px (0 \leq x \leq x_0)$; (a) 绕 Ox 轴; (6) 绕 Oy 轴.

解 (a) $\sqrt{1+y_x'^2} = \frac{\sqrt{p+2x}}{\sqrt{2x}},$

于是, 所求的表面积为

$$\begin{aligned} P_x &= 2\pi \int_0^{x_0} \sqrt{2px} \cdot \frac{\sqrt{p+2x}}{\sqrt{2x}} dx \\ &= \frac{2\pi}{3} \left[(2x_0 + p) \sqrt{2px_0 + p^2} - p^2 \right]. \end{aligned}$$

(6) $\sqrt{1+x_y'^2} = \frac{\sqrt{p^2+y^2}}{p}.$

于是, 所求的表面积为

$$\begin{aligned} P_y &= 4\pi \int_0^{\sqrt{2px_0}} x \sqrt{1+x_y'^2} dy \\ &= 4\pi \int_0^{\sqrt{2px_0}} \frac{y^2}{2p} \cdot \frac{\sqrt{p^2+y^2}}{p} dy \\ &= \frac{2\pi}{p^2} \int_0^{\sqrt{2px_0}} y^2 \sqrt{p^2+y^2} dy \\ &= \frac{2\pi}{p^2} \left[\frac{y(2y^2+p^2)}{8} \sqrt{p^2+y^2} \right. \\ &\quad \left. - \frac{p^4}{8} \ln(y + \sqrt{y^2+p^2}) \right] \Big|_0^{\sqrt{2px_0}} \\ &= \frac{\pi}{4} \left[(p+4x_0) \sqrt{2x_0(p+2x_0)} \right. \\ &\quad \left. - p^2 \ln \frac{\sqrt{2x_0} + \sqrt{p+2x_0}}{\sqrt{p}} \right]. \end{aligned}$$

2490. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (0 < b \leq a)$; (a) 绕 Ox 轴; (6) 绕 Oy 轴.

解 (a) $y^2 = b^2 - \frac{b^2}{a^2}x^2, yy' = -\frac{b^2}{a^2}x,$

$$\begin{aligned} y \sqrt{1+y'^2} &= \sqrt{y^2 + (yy')^2} \\ &= \frac{b}{a} \sqrt{a^2 - \frac{a^2 - b^2}{a^2}x^2} = \frac{b}{a} \sqrt{a^2 - \epsilon^2 x^2}. \end{aligned}$$

于是, 所求的表面积为

$$\begin{aligned} P_x &= 2\pi \frac{b}{a} \int_{-a}^a \sqrt{a^2 - \epsilon^2 x^2} dx \\ &= \frac{2\pi b}{a} \left(a \sqrt{a^2 - \epsilon^2 a^2} + \frac{a^2}{\epsilon} \arcsin \epsilon \right) \\ &= 2\pi b \left(b + \frac{a}{\epsilon} \arcsin \epsilon \right), \end{aligned}$$

其中 $\epsilon = \frac{\sqrt{a^2 - b^2}}{a}$ 是椭圆之离心率.

(6) 将 x, y 轴对调, 即将 x 轴作为短轴. 于是在所得出的 $y \sqrt{1+y'^2}$ 中仅需将 a 与 b 的位置对调一下即可, 即

$$\begin{aligned} y \sqrt{1+y'^2} &= \frac{a}{b} \sqrt{b^2 + \frac{a^2 - b^2}{b^2}x^2} \\ &= \frac{a}{b} \sqrt{b^2 + \frac{c^2}{b^2}x^2}. \end{aligned}$$

于是, 所求表面积为

$$\begin{aligned} P_y &= 2\pi \frac{a}{b} \int_{-b}^b \sqrt{b^2 + \frac{c^2}{b^2}x^2} dx \\ &= 2\pi a \frac{1}{b} \left[\frac{x}{2} \sqrt{b^2 + \frac{c^2}{b^2}x^2} \right. \\ &\quad \left. + \frac{b^3}{2c} \ln \left(\frac{c}{b}x + \sqrt{b^2 + \frac{c^2}{b^2}x^2} \right) \right] \Big|_{-b}^b \end{aligned}$$

2493. $y = a \operatorname{ch} \frac{x}{a}$ ($|x| \leq b$); (a) 绕 Ox 轴;

(6) 绕 Oy 轴.

解 (a) $\sqrt{y'^2 + 1} = \sqrt{\operatorname{sh}^2 \frac{x}{a} + 1} = \operatorname{ch} \frac{x}{a}$.

于是, 所求的表面积为

$$\begin{aligned} P_x &= 2\pi a \int_{-b}^b \operatorname{ch}^2 \frac{x}{a} dx \\ &= 2\pi a \int_0^b \left(1 + \operatorname{ch} \frac{2x}{a} \right) dx \\ &= \pi a \left(2b + a \operatorname{sh} \frac{2b}{a} \right). \end{aligned}$$

$$\begin{aligned} (6) \quad P_y &= 2\pi \int_0^b x \sqrt{1 + y'^2} dx \\ &= 2\pi \int_0^b x \operatorname{ch} \frac{x}{a} dx \\ &= 2\pi a \left(a + b \operatorname{sh} \frac{b}{a} - a \operatorname{ch} \frac{b}{a} \right). \end{aligned}$$

2494. $\pm x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$ 绕 Ox 轴.

解 $x'_y = \mp \frac{\sqrt{a^2 - y^2}}{y}$, $\sqrt{1 + x'^2_y} = \frac{a}{y}$
($0 \leq y \leq a$).

于是, 所求的表面积为

$$P_x = 2 \cdot 2\pi \int_0^a y \frac{a}{y} dy = 4\pi a^2.$$

2495. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$);

(a) 绕 Ox 轴; (6) 绕 Oy 轴; (B) 绕直线 $y = 2a$;

解 先求 ds :

$$ds = \sqrt{x_i'^2 + y_i'^2} dt = 2a \sin \frac{t}{2} dt.$$

于是, 所求的表面积为

$$\begin{aligned} \text{(a)} \quad P_x &= 2\pi \int_0^{2\pi} a(1 - \cos t) \cdot 2a \sin \frac{t}{2} dt \\ &= 16\pi a^2 \int_0^\pi \sin^3 u du = \frac{64}{3}\pi a^2. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P_x &= 2\pi \int_0^{2\pi} a(t - \sin t) \cdot 2a \sin \frac{t}{2} dt \\ &= 4\pi a^2 \int_0^{2\pi} (t - \sin t) \sin \frac{t}{2} dt = 16\pi^2 a^2. \end{aligned}$$

(B) 作平移 $x = \bar{x}, y = \bar{y} + 2a$ 则

$$\bar{y} = -a(1 + \cos t).$$

于是, 所求的表面积为

$$\begin{aligned} P_{\bar{x}} &= \left| 2\pi \int_0^{2\pi} [-a(1 + \cos t)] 2a \sin \frac{t}{2} dt \right|^{''} \\ &= \frac{32}{3}\pi a^2. \end{aligned}$$

*) 在此取绝对值, 是由于被积函数始终不为正之故.

2496. $x = a \cos^3 t, y = a \sin^3 t$ 绕直线 $y = x$.

解 先求 ds :

$$\begin{aligned} ds &= \sqrt{x_i'^2 + y_i'^2} dt \\ &= \begin{cases} 3a \sin t \cos t dt, & \text{当 } \frac{\pi}{4} \leq t \leq \frac{\pi}{2}, \\ -3a \sin t \cos t dt, & \text{当 } \frac{\pi}{2} \leq t \leq \frac{3\pi}{4}. \end{cases} \end{aligned}$$

利用对称性,并作旋转,即得所求的表面积为

$$\begin{aligned}
 P &= 2 \left(2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{yx}{\sqrt{2}} \sqrt{x'^2 + y'^2} dt \right. \\
 &\quad \left. + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{y-x}{\sqrt{2}} \sqrt{x'^2 + y'^2} dt \right) \\
 &= \frac{4\pi}{\sqrt{2}} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (a\sin^3 t - a\cos^3 t) \cdot 3a\sin t \cos t dt \right. \\
 &\quad \left. - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (a\sin^3 t - a\cos^3 t) \cdot 3a\sin t \cos t dt \right) \\
 &= \frac{12\pi a^2}{\sqrt{2}} \left(\left(\frac{1}{5} \sin^5 t + \frac{1}{5} \cos^5 t \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right. \\
 &\quad \left. - \left(\frac{1}{5} \sin^5 t + \frac{1}{5} \cos^5 t \right) \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \right) \\
 &= \frac{3}{5} \pi a^2 (4\sqrt{2} - 1).
 \end{aligned}$$

2497. $r = a(1 + \cos \varphi)$ 绕极轴.

解 $ds = \sqrt{r^2 + r'^2} d\varphi = 2a \cos \frac{\varphi}{2} d\varphi,$
 $y = r \sin \varphi = a(1 + \cos \varphi) \sin \varphi$
 $= 4a \cos^3 \frac{\varphi}{2} \sin \frac{\varphi}{2}.$

于是,所求的表面积为

$$P = 2\pi \int_0^\pi 8a^2 \cos^4 \frac{\varphi}{2} \sin \frac{\varphi}{2} d\varphi = \frac{32}{5} \pi a^2.$$

2498. $r^2 = a^2 \cos 2\varphi$: (a) 绕极轴; (b) 绕轴 $\varphi = \frac{\pi}{2}$;

(b) 绕轴 $\varphi = \frac{\pi}{4}$.

解 (a) $y = a \sqrt{\cos 2\varphi} \sin \varphi, ds = \frac{a}{\sqrt{\cos 2\varphi}} d\varphi.$

于是, 所求的表面积为

$$P = 2 \cdot 2\pi \int_0^{\frac{\pi}{4}} a^2 \sin \varphi d\varphi = 2\pi a^2 (2 - \sqrt{2}).$$

$$(b) \quad x = a \sqrt{\cos 2\varphi} \cos \varphi \left(-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4} \right).$$

于是, 所求的表面积为

$$\begin{aligned} P &= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a \sqrt{\cos 2\varphi} \cos \varphi \frac{a}{\sqrt{\cos 2\varphi}} d\varphi \\ &= 2\pi a^2 \sqrt{2}. \end{aligned}$$

$$(c) \quad x = a \sqrt{\cos 2\varphi} \cos \varphi, y = a \sqrt{\cos 2\varphi} \sin \varphi,$$

$$ds = \frac{a}{\sqrt{\cos 2\varphi}} d\varphi.$$

注意到在 $-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$ 内恒有 $x - y \geq 0$, 于是, 所求的表面积为

$$\begin{aligned} P &= 2 \cdot 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x - y}{\sqrt{2}} \frac{a}{\sqrt{\cos 2\varphi}} d\varphi \\ &= \frac{4\pi a^2}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos \varphi - \sin \varphi) d\varphi \\ &= \frac{4\pi a^2}{\sqrt{2}} (\sin \varphi + \cos \varphi) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= 4\pi a^2. \end{aligned}$$

2499. 由抛物线 $ay = a^2 - x^2$ 及 Ox 轴所包围的图形绕 Ox 轴旋转而构成一旋转体. 求其表面积与等体积球的表面积之比.

解 首先求此旋转体的表面积.

$$\sqrt{1+y'^2} = \frac{2\sqrt{x^2 + \frac{a^2}{4}}}{a},$$

从而

$$\begin{aligned} P_x &= 2 \cdot 2\pi \int_0^a \left(a - \frac{x^2}{a}\right) \cdot \frac{2\sqrt{x^2 + \frac{a^2}{4}}}{a} dx \\ &= 8\pi \int_0^a \sqrt{x^2 + \frac{a^2}{4}} dx - \frac{8\pi}{a^2} \int_0^a x^2 \sqrt{x^2 + \frac{a^2}{4}} dx \\ &= 8\pi \left\{ \frac{x}{2} \sqrt{x^2 + \frac{a^2}{4}} + \frac{a^2}{8} \ln \left[x + \sqrt{x^2 + \frac{a^2}{4}} \right] \right\} \Big|_0^a \\ &\quad - \frac{8\pi}{a^2} \left\{ \frac{x \left(2x^2 + \frac{a^2}{4} \right)}{8} \sqrt{x^2 + \frac{a^2}{4}} \right. \\ &\quad \left. - \frac{a^4}{128} \ln \left[x + \sqrt{x^2 + \frac{a^2}{4}} \right] \right\} \Big|_0^a \\ &= \frac{\pi a^2}{8} \left(7\sqrt{5} + \frac{17}{2} \ln(2 + \sqrt{5}) \right); \end{aligned}$$

其次, 求旋转体的体积.

$$V_x = \pi \int_{-a}^a \left(a - \frac{x^2}{a}\right)^2 dx = \frac{16\pi a^3}{15}.$$

设与其等体积球的半径为 R , 则

$$\frac{4\pi R^3}{3} = \frac{16\pi a^3}{15}.$$

所以

$$R = \sqrt[3]{\frac{4}{3}} a.$$

于是,此球的表面积为

$$P = 4\pi R^2 = 4\pi \sqrt[3]{\frac{16}{25}} a^2.$$

最后得到

$$\begin{aligned} \frac{P_1}{P} &= \frac{\frac{\pi a^2}{8} \left(7\sqrt{5} + \frac{17}{2} \ln(2 + \sqrt{5}) \right)}{\frac{8\pi a^2}{5} \sqrt[3]{10}} \\ &= \frac{5[14\sqrt{5} + 17\ln(2 + \sqrt{5})]}{128 \cdot \sqrt[3]{10}} \\ &\doteq 1.013. \end{aligned}$$

*) 利用 1820 题的结果.

2500. 由直线 $x = \frac{p}{2}$ 与抛物线 $y^2 = 2px$ 所包围的图形绕直线 $y = p$ 而旋转, 求这旋转体的体积和表面积.

$$\begin{aligned} \text{解 } V_{y=p} &= \int_0^{\frac{p}{2}} \pi(p + \sqrt{2px})^2 dx \\ &\quad - \int_0^{\frac{p}{2}} \pi(p - \sqrt{2px})^2 dx \\ &= 4\pi p \int_0^{\frac{p}{2}} \sqrt{2px} dx \\ &= \frac{4}{3} \pi p^3. \end{aligned}$$

旋转体的侧面积为

$$\begin{aligned} S_{\text{侧}} &= \int_{(G)} 2\pi(p + \sqrt{2px}) ds \\ &\quad + \int_{(G)} 2\pi(p - \sqrt{2px}) ds \\ &= 4\pi p \int_{(G)} ds = 4\pi p \int_0^p \sqrt{1 + \frac{y^2}{p^2}} dy \end{aligned}$$

$$\begin{aligned}
&= 4\pi \int_0^p \sqrt{y^2 + p^2} dy \\
&= 4\pi \left(\frac{y}{2} \sqrt{y^2 + p^2} + \frac{p^2}{2} \ln(y + \sqrt{y^2 + p^2}) \right) \Big|_0^p \\
&= 2\pi p^2 [\sqrt{2} + \ln(1 + \sqrt{2})],
\end{aligned}$$

而底面积为

$$S_{\text{底}} = \pi(2p)^2 = 4\pi p^2,$$

于是,所求的表面积为

$$\begin{aligned}
P &= S_{\text{侧}} + S_{\text{底}} \\
&= 2\pi p^2 [(2 + \sqrt{2}) + \ln(1 + \sqrt{2})].
\end{aligned}$$

§ 9. 矩的计算法. 重心的坐标

1° 矩 若在 Oxy 平面上,密度为 $\rho = \rho(y)$ 的质量 M 充满了某有界连续统 Ω (曲线,平面的区域),而 $\omega = \omega(y)$ 为 Ω 中纵标不超过 y 的部分的对应的度量(弧长,面积),则数

$$\begin{aligned}
M_k &= \lim_{\max |\Delta y_i| \rightarrow 0} \sum_{i=1}^n \rho(y_i) y_i^k \Delta \omega(y_i) \\
&= \int_{\Omega} \rho y^k d\omega(y) \quad (k = 0, 1, 2, \dots)
\end{aligned}$$

称为质量 M 对于 Ox 轴的 k 次矩.

特殊情形,当 $k = 0$ 时得质量 M ,当 $k = 1$ 时得静力矩,当 $k = 2$ 时得转动惯量.

同样地可定义出质量对于坐标平面的矩.

若 $\rho = 1$,则对应的矩称为几何矩(线矩,面积矩,体积矩等等).

2° 重心 均匀平面图形 S 的重心的坐标 (x_0, y_0) 根据下面的公式来定义

$$x_0 = \frac{M_1^{(y)}}{S}, y_0 = \frac{M_1^{(x)}}{S},$$

式中 $M_1^{(y)}, M_1^{(x)}$ 为面积 S 对于 Oy 轴和 Ox 轴的几何静力矩.

2501. 求半径为 a 的半圆弧对于过此弧两端点直径的静力矩和转动惯量.

解 取此直径所在的直线作为 Ox 轴, 圆心作为原点, 则圆的方程为

$$x^2 + y^2 = a^2.$$

从而

$$y = \sqrt{a^2 - x^2}$$

及

$$ds = \sqrt{1 + y'^2} dx = \frac{a}{y} dx = \frac{a}{\sqrt{a^2 - x^2}} dx.$$

于是, 所求的静力矩和转动惯量^{*} 为

$$M_1 = \int_{-a}^a \sqrt{a^2 - x^2} \cdot \frac{a}{\sqrt{a^2 - x^2}} dx = 2a^2$$

及

$$\begin{aligned} M_2 &= \int_{-a}^a (a^2 - x^2) \cdot \frac{a}{\sqrt{a^2 - x^2}} dx \\ &= 2a \int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi a^3}{2}. \end{aligned}$$

2502. 求底为 b , 高为 h 的均匀三角形薄板对于其底边的静力矩和转动惯量 ($\rho = 1$).

解 取坐标系如图 4.46 所示.

$$M_1^{(x)} = \frac{1}{2} \int_0^b y^2 dx = \frac{1}{2} \int_0^c y_1^2 dx + \frac{1}{2} \int_c^b y_2^2 dx.$$

由于

*) 这里假定 $\rho = 1$, 今后有类似情况, 不再说明.

$$y_1 = y_1(x) = \frac{h}{c}x,$$

$$\begin{aligned} y_2 &= y_2(x) \\ &= \frac{h}{c-b}(x-b), \end{aligned}$$

于是,所求的静力矩为

$$\begin{aligned} M_1^{(x)} &= \frac{1}{2} \int_0^c \frac{h^2}{c^2} x^2 dx \\ &+ \frac{1}{2} \int_c^b \frac{h^2}{(c-b)^2} (x-b)^2 dx = \frac{bh^2}{6}. \end{aligned}$$

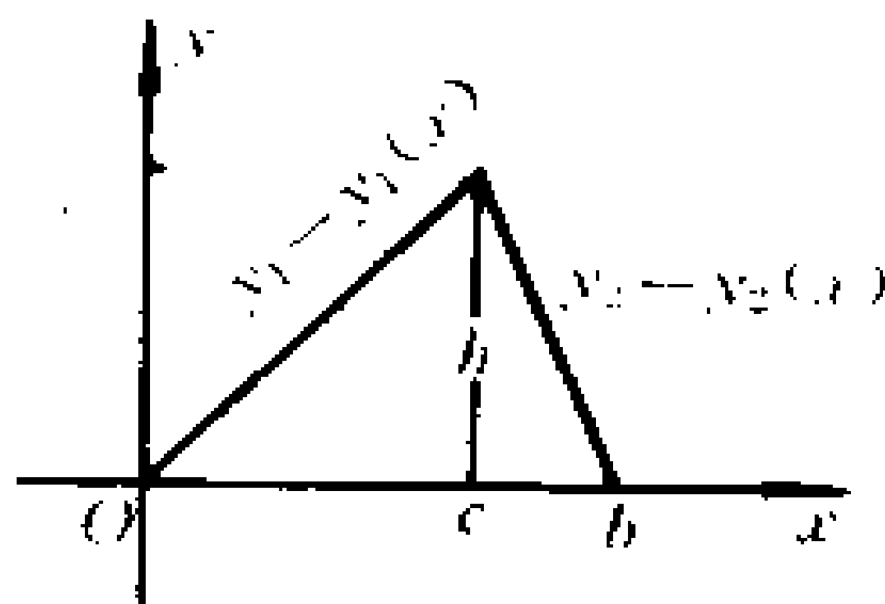


图 4.46

又由于

$$x_1 = x_1(y) = \frac{c}{h}y,$$

$$x_2 = x_2(y) = b + \frac{c-b}{h}y,$$

于是,所求的转动惯量为

$$\begin{aligned} M_2^{(x)} &= \int_0^h y^2 (x_2 - x_1) dy \\ &= \int_0^h y^2 \left(b - \frac{b}{h}y \right) dy = \frac{bh^3}{12}. \end{aligned}$$

2503. 求半轴长为 a 和 b 的均匀椭圆形薄板对于其主轴的转动惯量($\rho = 1$).

解 不妨设椭圆的方程为

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

则上、下半椭圆方程为

$$x_1 = -\frac{a}{b} \sqrt{b^2 - y^2},$$

$$x_2 = \frac{a}{b} \sqrt{b^2 - y^2}.$$

$$M_1 = \int_0^h x \cdot P(x) dx,$$

其中

$$P(x) = \pi y^2 = \pi \left[\frac{r}{h} (h - x) \right]^2.$$

于是,所求的静力矩和转动惯量分别为

$$M_1 = \frac{\pi r^2}{h^2} \int_0^h x (h - x)^2 dx = \frac{\pi r^2 h^2}{12};$$

$$\begin{aligned} M_2 &= \int_0^h x^2 \cdot P(x) dx \\ &= \frac{\pi r^2}{h^2} \int_0^h x^2 (h - x)^2 dx = \frac{\pi r^2 h^3}{30}. \end{aligned}$$

2505. 证明古尔金第一定理:弧 C 绕着不与它相交的轴旋转而成的旋转面的面积,等于这个弧的长度与这弧的重心所划出的圆周之长的乘积.

证 重心 (ξ, η) 具有这样的性质,即如把曲线的全部“质量”都集中到它上面,则此质量对于任何一个轴的静力矩,都与曲线对此轴的静力矩相同.即

$$\xi s = M_y = \int_0^s x ds,$$

$$\eta s = M_x = \int_0^s y ds,$$

式中 s 表示弧长.

于是

$$2\pi\eta \cdot s = 2\pi \int_0^s y ds.$$

上式的右端是弧 C 旋转而成的曲面面积,左端 $2\pi\eta$ 表示弧 C 绕 Ox 轴旋转时其重心所划出的圆周之长.从而定理得证.

2506. 证明古尔金第二定理: 面积 S 绕不与它相交的轴旋转而成的旋转体, 其体积等于面积 S 与这面积的重心所划出的圆周之长的相乘积.

证 由于

$$\eta \cdot S = M_y = \frac{1}{2} \int_a^b y^2 dx,$$

所以

$$2\pi\eta \cdot S = \pi \int_a^b y^2 dx.$$

上式右端即为旋转体的体积, 从而定理得证.

2507. 求圆弧: $x = a\cos\varphi, y = a\sin\varphi$ ($|\varphi| \leq \alpha \leq \pi$) 重心的坐标.

解 显见

$$\eta = 0,$$

圆弧长

$$s = 2a\alpha.$$

由于

$$M_y = \int_0^s x ds = \int_{-\alpha}^{\alpha} a^2 \cos\varphi d\varphi = 2a^2 \sin\alpha,$$

所以

$$\xi = \frac{2a^2 \sin\alpha}{2a\alpha} = \frac{a \sin\alpha}{\alpha}.$$

即重心为 $\left(\frac{a \sin\alpha}{\alpha}, 0\right)$.

2508. 求抛物线: $ax = y^2, ay = x^2$ ($a > 0$) 所围成面积的重心的坐标.

解 利用古尔金第二定理来解此题. 首先, 此面积为

$$S = \frac{a^2}{3},$$

体积为

$$V = \pi \int_0^a \left(ax - \frac{x^4}{a^2} \right) dx = \frac{3\pi a^3}{10}.$$

于是

$$2\pi\eta \cdot \frac{a^2}{3} = \frac{3\pi a^3}{10},$$

所以

$$\eta = \frac{9a}{20}.$$

利用对称性知

$$\xi = \eta = \frac{9a}{20}.$$

即所求的重心为 $\left(\frac{9a}{20}, \frac{9a}{20} \right)$.

*) 利用 2397 题的结果.

2509. 求面积

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad (0 \leq x \leq a, 0 \leq y \leq b)$$

的重心的坐标.

解 首先, 我们已知第一象限椭圆的面积等于 $\frac{\pi ab}{4}$.

其次, 我们再求椭圆绕 Ox 轴旋转所得的旋转体体积. 因为

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

所以

$$V = \pi \int_{-a}^a \frac{b^2}{a^2}(a^2 - x^2) dx = \frac{4}{3} \pi ab^2.$$

按古尔金第二定理,我们有

$$2\pi\eta \frac{\pi ab}{4} = \frac{2}{3}\pi ab^2,$$

所以

$$\eta = \frac{4b}{3\pi}.$$

同理可求得

$$\xi = \frac{4a}{3\pi}.$$

事实上,只须在结果中将 a 和 b 对调即得.于是,所求的

重心为 $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.

2510. 求半径为 a 的均匀半球的重心坐标.

解 取圆心作为原点,则球的方程为

$$x^2 + y^2 + z^2 = a^2.$$

设重心为 (ξ, η, ζ) ,显见 $\xi = \eta = 0$.而

$$V_{\text{半球}} = \frac{2\pi a^3}{3}.$$

将圆

$$y^2 + z^2 = a^2$$

绕 Oz 轴旋转,即得球.

又

$$\begin{aligned} M_1^{(x)} &= \int_{(V)} z dV = \pi \int_0^a z y^2 dz \\ &= \pi \int_0^a z (a^2 - z^2) dz = \frac{\pi a^4}{4}. \end{aligned}$$

最后得到

$$\zeta = \frac{M_1^{(z)}}{V} = \frac{\frac{\pi a^4}{4}}{\frac{2\pi a^3}{3}} = \frac{3a}{8}.$$

于是, 所求重心为 $\left(0, 0, \frac{3a}{8}\right)$.

2511. 求对数螺线

$$r = ae^{m\varphi} \quad (m > 0)$$

上由点 $O(-\infty, 0)$ 到点 $P(\varphi, r)$ 的弧 OP 的重心 $C(\varphi_0, r_0)$ 之坐标. 当 P 点移动时, C 点画出怎样的曲线?

解 重心的直角坐标为

$$\begin{aligned} \xi &= \frac{\int_{(1)} x ds}{\int_{(1)} ds} \\ &= \frac{\int_{-\infty}^{\varphi} r \cos \varphi \cdot \sqrt{a^2(1+m^2)} e^{m\varphi} d\varphi}{\int_{-\infty}^{\varphi} \sqrt{a^2(1+m^2)} e^{m\varphi} d\varphi} \\ &= \frac{a \int_{-\infty}^{\varphi} e^{2m\varphi} \cos \varphi d\varphi}{\int_{-\infty}^{\varphi} e^{m\varphi} d\varphi} \\ &= \frac{mae^{m\varphi}(\sin \varphi + 2m \cos \varphi)}{4m^2 + 1}. \end{aligned}$$

同法可得

$$\eta = \frac{\int_{(1)} y ds}{\int_{(1)} ds} = \frac{mae^{m\varphi}(2m \sin \varphi - \cos \varphi)}{4m^2 + 1}.$$

于是, 重心的极坐标为

$$r_0 = \sqrt{\xi^2 + \eta^2} = \frac{ma}{4m^2 + 1} \sqrt{4m^2 + 1} e^{m\varphi}$$

$$= \frac{mr}{\sqrt{4m^2 + 1}},$$

$$\operatorname{tg} \varphi_0 = \frac{\eta}{\xi} = \frac{2m \operatorname{tg} \varphi - 1}{\operatorname{tg} \varphi + 2m} = \frac{\operatorname{tg} \varphi - \frac{1}{2m}}{1 + \frac{1}{2m} \operatorname{tg} \varphi}$$

即 $\varphi_0 = \varphi - \alpha$, 其中 $\alpha = \arctan \frac{1}{2m}$.

当 P 点移动时, $C(\varphi_0, r_0)$ 画出的曲线为

$$r_0 = \frac{ma}{\sqrt{4m^2 + 1}} e^{m\varphi} = \frac{ma}{\sqrt{4m^2 + 1}} e^{m(\varphi_0 + \alpha)}.$$

这也是一条对数螺线.

2512. 求曲线 $r = a(1 + \cos \varphi)$ 所围面积的重心坐标.

解 计算时, 将小扇形的重量集中在其重心

$\left(\frac{2}{3} r \cos \varphi, \frac{2}{3} r \sin \varphi \right)$ 处. 由对称性知 $\eta = 0$, 而

$$\xi = \frac{\int_{(G)} xy dx}{\int_{(G)} y dx}$$

$$= \frac{\frac{2}{3} \int_0^\pi r \cos \varphi \cdot \frac{1}{2} r^2 d\varphi}{\int_0^\pi \frac{1}{2} r^2 d\varphi}$$

$$= \frac{2}{3} \frac{\int_0^\pi a^3 (1 + \cos \varphi)^3 \cos \varphi d\varphi}{\int_0^\pi a^2 (1 + \cos \varphi)^2 d\varphi}$$

$$\begin{aligned}
&= \frac{2a}{3} \frac{\int_0^\pi (1 + 3\cos\varphi + 3\cos^2\varphi + \cos^3\varphi)\cos\varphi d\varphi}{\int_0^\pi (1 + 2\cos\varphi + \cos^2\varphi)d\varphi} \\
&= \frac{5a}{6}.
\end{aligned}$$

于是,重心的极坐标为 $\varphi_0 = 0, r_0 = \frac{5a}{6}$.

2513. 求摆线 $x = a(t - \sin t), y = a(1 - \cos t) (0 \leq t \leq 2\pi)$ 的第一拱与 Ox 轴所围成面积的重心的坐标.

解 由对称性知 $\xi = \pi a$. 由于面积 $S = 3\pi a^2$ *) 及面积 S 绕 Ox 轴旋转而成的曲面包围的体积 $V_x = 5\pi^2 a^3$ **), 利用古尔金第二定理, 即得重心 (ξ, η) 适合下列关系式

$$2\pi\eta \cdot S = V_x$$

或

$$\eta = \frac{V_x}{2\pi S} = \frac{5\pi^2 a^3}{2\pi \cdot 3\pi a^2} = \frac{5a}{6}.$$

于是,重心为 $\left(\pi a, \frac{5a}{6}\right)$.

*) 利用 2413 题的结果.

**) 利用 2480 题(a) 的结果.

***) 参看 2506 题.

2514. 求面积 $0 \leq x \leq a, y^2 \leq 2px$ 绕 Ox 轴旋转所成旋转体的重心的坐标.

解 由对称性知 $\eta = 0$. 又

$$\xi = \frac{\int_0^a x\pi y^2 dx}{\int_0^a \pi y^2 dx} = \frac{\int_0^a 2px^2 dx}{\int_0^a 2p x dx}$$

$$= \frac{2}{3}a.$$

于是,所求的重心为 $\left(\frac{2}{3}a, 0\right)$.

2515. 求半球 $x^2 + y^2 + z^2 = a^2 (z \geq 0)$ 的重心的坐标.

解 由对称性知

$$\xi = \eta = 0.$$

$$\begin{aligned} \zeta &= \frac{\int_0^a 2 \cdot 2\pi x \sqrt{1+x'^2} dz}{\int_0^a 2\pi x \sqrt{1+x'^2} dz} \\ &= \frac{\int_0^a 2\pi z \sqrt{a^2 - z^2} \cdot \frac{a}{\sqrt{a^2 - z^2}} dz}{\int_0^a 2\pi \sqrt{a^2 - z^2} \cdot \frac{a}{\sqrt{a^2 - z^2}} dz} \\ &= \frac{2\pi a \int_0^a z dz}{2\pi a \int_0^a dz} = \frac{2\pi a \cdot \frac{1}{2}a^2}{2\pi a^2} = \frac{a}{2}. \end{aligned}$$

于是,所求的重心为 $\left(0, 0, \frac{a}{2}\right)$.

*) 在此是将 $x^2 + z^2 = a^2$ 绕 Oz 轴旋转而得半球面.

§ 10. 力学和物理学中的问题

作成适当的积分和并找出它们的极限,来解下列问题:

2516. 轴的长度 $l = 10$ 米,若该轴的线性密度按定律 $\delta = 6 + 0.3x$ 千克/米而变更,其中 x 为距轴两端点中之一

端的距离,求轴的质量.

解 将轴 n 等分,每份的长 $\Delta x = \frac{10}{n}$. 把每小段近似地看成是均匀的,并以右端点的密度作为小段的密度. 这样,便得到轴的质量 M 的近似值,即

$$M \approx \sum_{i=1}^n \left(6 + 0.3 \times \frac{10}{n} i \right) \frac{10}{n}.$$

显然, n 愈大愈近似,于是,得轴的质量

$$\begin{aligned} M &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 + 0.3 \times \frac{10}{n} i \right) \frac{10}{n} \\ &= \lim_{n \rightarrow \infty} \left[60 + \frac{15 \times (n+1)}{n} \right] = 75 (\text{千克}). \end{aligned}$$

2517. 把质量为 m 的物体从地球(其半径为 R) 表面升高到高度为 h 的地位,需要化费多大的功?若物体远离至无穷远去,则功等于什么?

解 由牛顿万有引力定律

$$f = k \frac{mM}{r^2},$$

其中 M 为地球的质量, r 为物体离开地球中心的距离, k 为比例常数. 将 h 分成 n 等份,在每份上把引力近似地看作是不变的,在第 i 份上取

$$r_i = \sqrt{\left[\frac{h}{n} (i-1) + R \right] \left[\frac{h}{n} i + R \right]}, \text{ 则力}$$

$$f_i = k \frac{mM}{\left[\frac{h}{n} (i-1) + R \right] \cdot \left[\frac{h}{n} i + R \right]}.$$

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{10}{n} \cdot \frac{10}{n} = \lim_{n \rightarrow \infty} 50 \frac{n+1}{n} \\ = 50(\text{千克厘米}) = 0.5(\text{千克米}).$$

2519. 直径为 20 厘米, 长为 80 厘米的圆柱被压强为 10 千克/厘米² 的蒸汽充满着. 假定气体的温度不变, 要使气体的体积减小一半, 须要花费多大的功?

解 由波义耳—马利奥特定律有

$$pv = C,$$

其中 p 表示气体的压强, v 表示体积, C 为常量. 由条件知, 常量

$$C = 10 \cdot \pi \cdot 100 \cdot 80 = 800\pi(\text{千克米}).$$

设初始时气体体积为 v_0 , 将区间 $\left[\frac{v_0}{2}, v_0\right]$ 分成 n 个小区间, 分点依次为

$$\frac{v_0}{2}, \frac{v_0}{2}q, \frac{v_0}{2}q^2, \dots, \frac{v_0}{2}q^i, \dots, \frac{v_0}{2}q^n = v_0,$$

其中 $q = \sqrt[n]{\frac{v_0}{\frac{v_0}{2}}} = \sqrt[n]{2}$. 由于气体体积从 $\frac{v_0}{2}q^{i+1}$ 减小

至 $\frac{v_0}{2}q^i$ 须要花费功的近似值为

$$C\left(\frac{v_0}{2}q^i\right)^{-1}\left(\frac{v_0}{2}q^{i+1} - \frac{v_0}{2}q^i\right),$$

于是, 所要求的功

$$W = \lim_{n \rightarrow \infty} \sum_{i=0}^n C\left(\frac{v_0}{2}q^i\right)^{-1}\left(\frac{v_0}{2}q^{i+1} - \frac{v_0}{2}q^i\right) \\ = \lim_{n \rightarrow \infty} Cn(\sqrt[n]{2} - 1) = C\ln 2^*)$$

$$= 800\pi \cdot \ln 2 \approx 1742 (\text{千克米}).$$

*) 利用 541 题的结果.

2520. 求水对于垂直壁上的压力, 这壁的形状为半圆形, 半径为 a 且其直径位于水的表面上.

解 为求出水对半圆形的压力, 只要计算出作用于四分之一圆上的压力, 然后再把它两倍起来. 现将四分之一圆等分成 n 个圆心角为 $\Delta\theta$ 的小扇形(图 4.48). 作用于该小扇形上的压力的近似值为

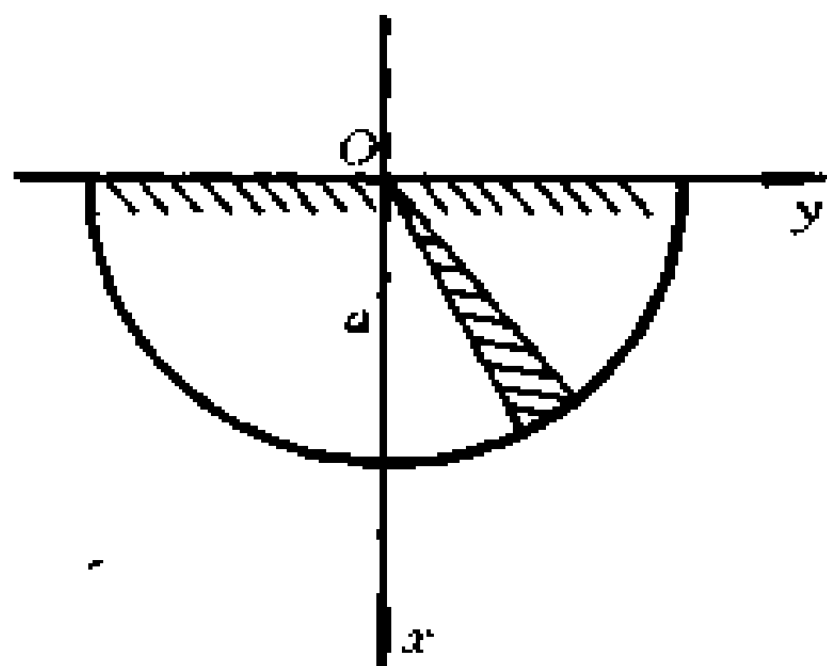


图 4.48

$$\frac{1}{2}a^2\Delta\theta \cdot \frac{2}{3}a\sin\theta_i,$$

其中 $\Delta\theta = \frac{\pi}{2n}$, $\theta_i = \frac{i\pi}{2n}$.

于是, 作用于半圆上的压力

$$\begin{aligned} P &= 2 \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2}a^2 \cdot \frac{2}{3}a\sin \frac{i\pi}{2n} \cdot \frac{\pi}{2n} \\ &= \frac{2a^3}{3} \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{i\pi}{2n} \cdot \frac{\pi}{2n} = \frac{2a^3}{3}. \end{aligned}$$

*) 利用 2187 题的结果.

2521. 求水对于垂直壁上的压力, 这壁的形状为梯形, 其下底 $a = 10$ 米, 上底 $b = 6$ 米, 高 $h = 5$ 米, 下底沉没于水面下的距离为 $c = 20$ 米.

解 取坐标系如图 4.49 所示. AB 所满足的方程为

$$y = \frac{4}{5}x - 6.$$

将区间 $[15, 20]n$ 等分, 每份长 $\Delta x = \frac{5}{n}$. 对应于 Δx 的小条上所受的压力的近似值为

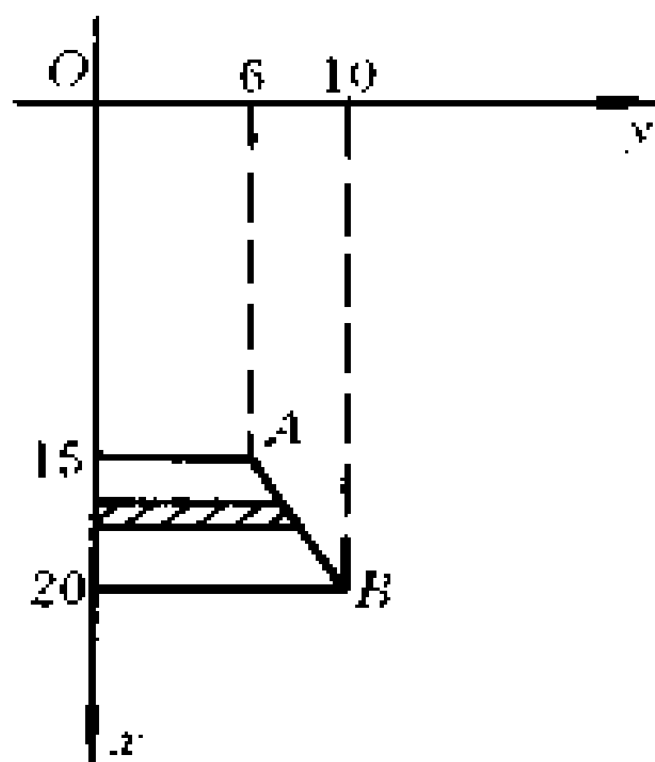


图 4.49

$$\left[\frac{4}{5} \left(15 + \frac{5i}{n} \right) - 6 \right] \left(15 + \frac{5i}{n} \right) \frac{5}{n}.$$

于是, 所要求的压力

$$\begin{aligned} P &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{4}{5} \left(15 + \frac{5i}{n} \right) - 6 \right] \left(15 + \frac{5i}{n} \right) \frac{5}{n} \\ &= 708 \frac{1}{3} (\text{吨})^* \end{aligned}$$

*) 仿照 2185 题和 2518 题的作法.

作出微分方程式以解下列问题:

2522. 点运动的速度是按下面的规律而变化:

$$v = v_0 + at.$$

问在闭间隔 $(0, T)$ 内这点经过的路程怎样?

解 设路程为 s , 则由导数的力学意义知

$$\frac{ds}{dt} = v = v_0 + at.$$

即 at 时间内经历的路程

$$ds = (v_0 + at)dt,$$

于是,

$$\begin{aligned}s &= \int_0^T (v_0 + at) dt \\ &= v_0 T + \frac{1}{2} a T^2.\end{aligned}$$

2523. 半径为 R 而密度为 δ 的均匀球体以角速度 ω 绕其直径而旋转. 求此球的动能.

解 已知半径为 R 质量为 M 的盘绕垂直盘心的轴的转动惯量为 $\frac{1}{2}MR^2$. 不妨设球面方程为 $x^2 + y^2 + z^2 = R^2$, 则考察以 dz 为厚度的垂直于 z 轴的圆盘, 其转动惯量为

$$\begin{aligned}dJ_z &= \frac{1}{2} \pi (R^2 - z^2) \delta \cdot (R^2 - z^2) dz \\ &= \frac{1}{2} \pi \delta (R^2 - z^2)^2 dz.\end{aligned}$$

从而球体的转动惯量

$$J_z = \int_{-R}^R \frac{1}{2} \pi \delta (R^2 - z^2)^2 dz = \frac{8}{15} \pi \delta R^5.$$

于是, 球的动能

$$E = \frac{1}{2} J \omega^2 = \frac{4}{15} \pi \delta \omega^2 R^5.$$

注 原题误为球壳, 现根据原答案予以改正.

2524. 具不变的线性密度 μ_0 的无穷直线以怎样的力吸引距此直线距离为 a 质量为 m 的质点?

解 取坐标系如图 4.50 所示, $|AO| = a$. 设引力在坐标轴上的射影为 F_x, F_y . 由于

$$dF_y = k \frac{m \mu_0 dx}{(a^2 + x^2)} \cos \varphi$$

$$= - \frac{km\mu_0 a}{(a^2 + x^2)^{\frac{3}{2}}} dx,$$

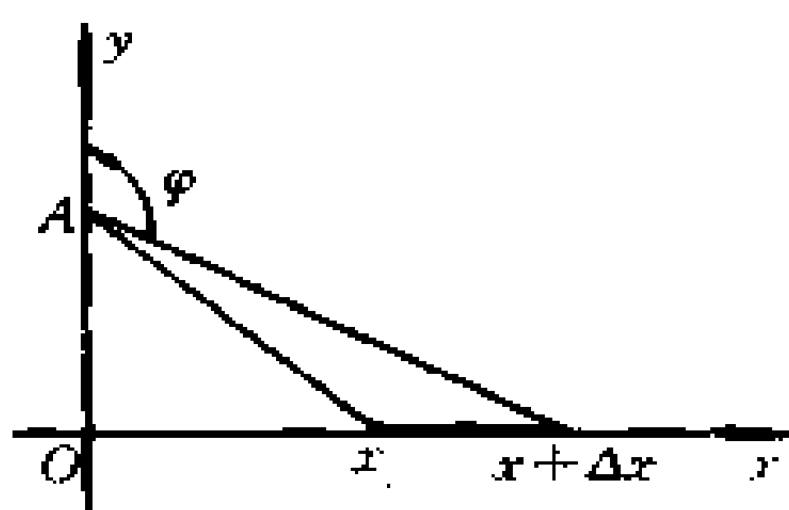


图 4.50

于是,

$$\begin{aligned} F_y &= - 2km\mu_0 a \int_0^{+\infty} \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} \\ &= - 2km\mu_0 a \cdot \frac{x}{a^2 \sqrt{a^2 + x^2}} \Big|_0^{+\infty} \\ &= - \frac{2km\mu_0}{a}. \end{aligned}$$

由对称性知, $F_x = 0$. 事实上, 我们有

$$\begin{aligned} F_x &= \int_{-\infty}^{+\infty} \frac{km\mu_0 \sin\varphi}{a^2 + x^2} dx \\ &= km\mu_0 \int_{-\infty}^{+\infty} \frac{x}{(a^2 + x^2)^{\frac{3}{2}}} dx = 0. \end{aligned}$$

其中 k 为引力常数. 由上述分析知, 引力指向 y 轴的负向.

2525. 计算半径为 a 及固定的表面密度为 δ_0 的圆形薄板以怎样的力吸引质量为 m 的质点 P , 此质点位于通过薄板中心 Q 且垂直于薄板平面的垂直线上, 最短距离

PQ 等于 b .

解 取坐标系如图 4.51 所示, 显然, 引力指向 y 轴的正向. 对于以 x 为半径的圆环, 其质量为 $dm = \delta_0 \cdot 2\pi x dx$, 对质点 P 的引力

$$\begin{aligned} dF_y &= 2km\delta_0\pi \frac{\cos\theta}{b^2 + x^2} dx \\ &= 2km\delta_0\pi \frac{bx}{(b^2 + x^2)^{\frac{3}{2}}} dx, \end{aligned}$$

于是, 所要求的引力

$$\begin{aligned} F_y &= 2km\delta_0\pi \int_0^a \frac{bx}{(b^2 + x^2)^{\frac{3}{2}}} dx \\ &= 2km\delta_0\pi \left(1 - \frac{b}{\sqrt{a^2 + b^2}} \right). \end{aligned}$$

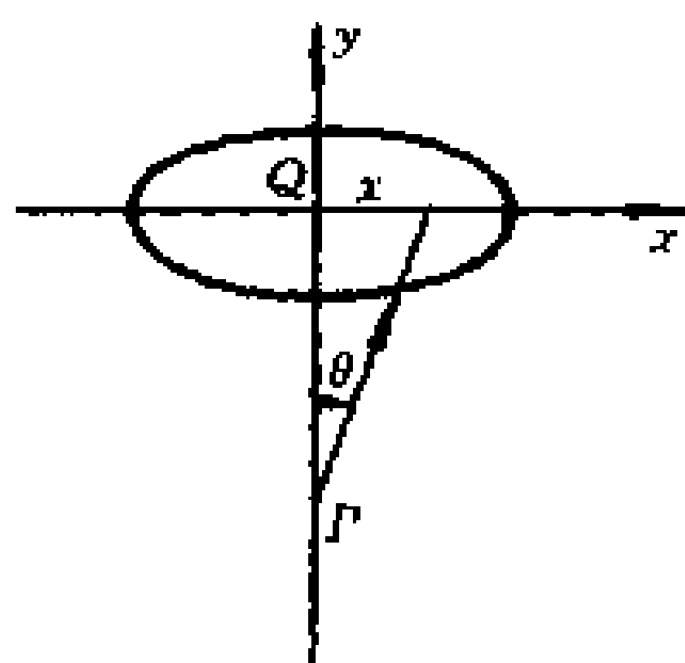


图 4.51

2526. 根据托里拆利定律, 液体从容器中流出的速度等于

$$v = c \sqrt{2gh},$$

式中 g 为重力加速度, h 为液体表面在开孔上之高, $c = 0.6$ 为实验系数.

直径为 $D = 1$ 米及高为 $H = 2$ 米的直立圆柱形

大桶, 充满之后从其底上直径为 $d = 1$ 厘米的圆孔流出, 须要多长时间, 完全流空?

解 取坐标系如图 4.52 所示. 对于 dt 时间, 从圆孔流出的液体体积 $dv = 0.15\pi \sqrt{2gx}dt$, 而桶内液体体积的减少量为

$dv = -\pi(50)^2 dx$, 其中 x 随时间 t 的增大而减小. 流出的量应等于桶内减少的量, 于是

$$-0.15\pi \sqrt{2gx}dt = \pi(50)^2 dx.$$

积分, 得

$$\int_0^t dt = - \int_{200}^x \frac{2500}{0.15} \frac{dx}{\sqrt{2gx}},$$

即

$$t = -33333 \frac{1}{\sqrt{2g}} (\sqrt{x} - \sqrt{200}),$$

其中 $g = 980$ 厘米/秒². 当 $x = 0$ 时, t 表示水流完所需的时间. 因而所要求的时间

$$t = \frac{33333 \sqrt{200}}{\sqrt{2 \times 980}} = 10648 (\text{秒}).$$

2527. 旋转体的容器应当是什么形状, 才能使液体流出时, 液体表面的下降是均匀的?

解 取坐标系如图 4.53 所示. 不妨设流出孔的半径

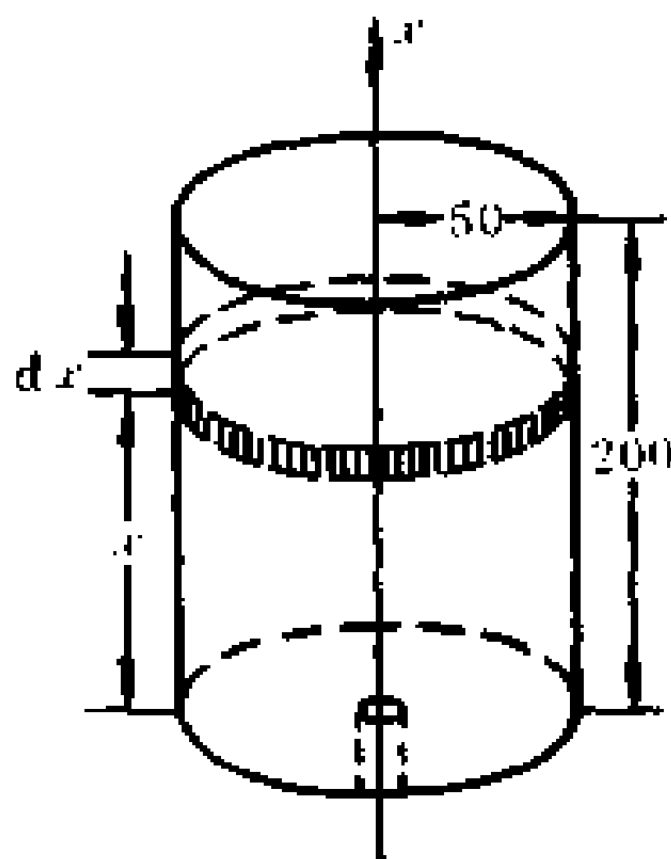


图 4.52

为单位厘米.

仿上题分析,得

$$\begin{aligned}\pi x^2 dy &= -\pi v dt \\ &= -\pi c \sqrt{2gy} dt,\end{aligned}$$

即

$$\begin{aligned}dy &= -c \sqrt{2g} \\ &\cdot \frac{\sqrt{y}}{x^2} dt.\end{aligned}$$

其中 c 为实验系数, g 为重力加速度.

由题意知

$$\frac{dy}{dt} = -c \sqrt{2g} \frac{\sqrt{y}}{x^2}$$

应等于常数 k , 即

$$-c \sqrt{2g} \frac{\sqrt{y}}{x^2} = k,$$

于是

$$y = Cx^4,$$

其中 C 为常数. 所以, 容器应当是把曲线 $y = Cx^4$ 绕铅直轴 Oy 旋转而得的曲面所构成的.

2528. 镭在每一时刻的分解速度与其现存的数量成比例, 设在开始的时刻 $t = 0$ 有镭 Q_0 克, 经过时间 $T = 1600$ 年它的量减少了一半. 求镭分解的规律.

解 设 Q 为镭现存的数量, 按题设有

$$\frac{dQ}{dt} = kQ,$$

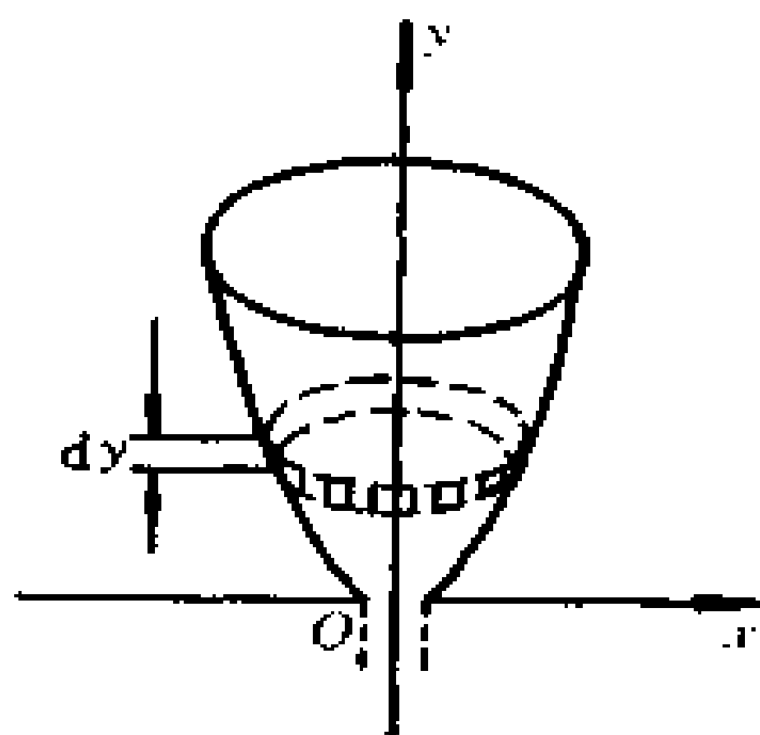


图 4.53

其中 k 为比例系数, 即

$$\frac{dQ}{Q} = kdt,$$

两端积分

$$\int_{Q_0}^{\frac{Q_0}{2}} \frac{dQ}{Q} = \int_0^{1600} kdt,$$

从而

$$k = -\frac{\ln 2}{1600}.$$

于是

$$\int_{Q_0}^Q \frac{dQ}{Q} = -\frac{\ln 2}{1600} \int_0^t dt,$$
$$\ln \frac{Q}{Q_0} = \ln 2^{-\frac{t}{1600}},$$

所以, 镭的分解规律为

$$Q = Q_0 \cdot 2^{-\frac{t}{1600}}.$$

2529⁺. 变换物质 A 为物质 B 的二阶化学反应之速度与此二物质的浓度相乘之积成正比. 问经过 $t = 1$ 小时在容器中所含有的物质 B 之百分率如何? 设 $t = 0$ 分时有 20% 的物质 B , 而当 $t = 15$ 分它变成 80%.

解 设 x 为生成物 B 的浓度, 按题设有

$$\frac{dx}{dt} = kx(1-x),$$

其中 k 为比例常数, 即

$$\frac{dx}{x(1-x)} = kdt.$$

两端积分

$$\begin{aligned}
&= \frac{\frac{1}{3}\pi r^2(H-h)\gamma}{\pi r^2 E} \\
&= \frac{1}{3} \frac{(H-h)\gamma}{E},
\end{aligned}$$

即

$$dl = \frac{1}{3} \frac{(H-h)\gamma}{E} dh.$$

于是圆锥形重棒总的伸长量为

$$\begin{aligned}
l &= \int_0^H \frac{1}{3} \frac{(H-h)\gamma}{E} dh \\
&= \frac{\gamma H^2}{6E}.
\end{aligned}$$

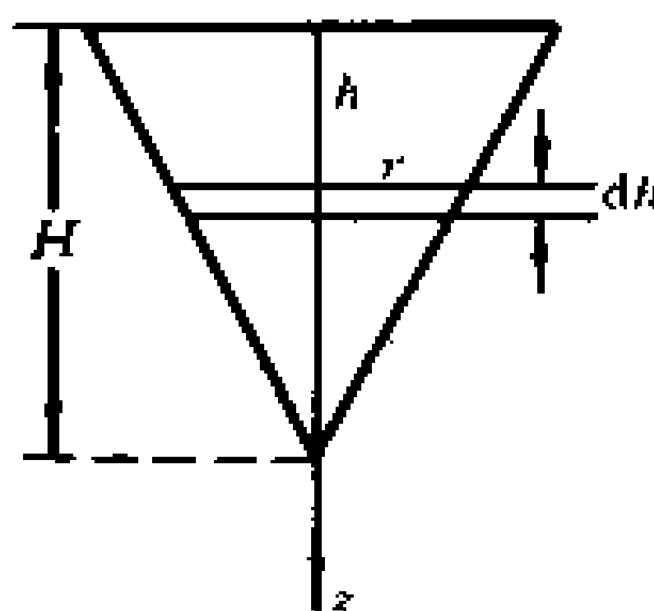


图 4.54

§ 11. 定积分的近似计算法

1° 矩形公式 若函数 $y = y(x)$ 于有穷的闭区间 $[a, b]$ 上连续且可微分充分多次数, 并且 $h = \frac{b-a}{n}$, $x_i = a + ih$ ($i = 0, 1, \dots, n$), $y_i = y(x_i)$, 则

$$\int_a^b y(x) dx = h(y_0 + y_1 + \dots + y_{n-1}) + R_n,$$

式中

$$R_n = \frac{(b-a)^2}{2n} y'(\xi) \quad (a \leq \xi \leq b).$$

2° 梯形公式 用相同的记号有

$$\begin{aligned}
\int_a^b y(x) dx &= h \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right) \\
&\quad + R_n,
\end{aligned}$$

式中

$$R_n = -\frac{(b-a)^3}{12n^2} f''(\xi') (a \leq \xi' \leq b).$$

3° 抛物线公式(辛普森公式) 命 $n = 2k$, 得

$$\begin{aligned} \int_a^b y(x) dx = \frac{h}{3} [& (y_0 + y_{2k}) + 4(y_1 + y_3 + \cdots \\ & + y_{2k-1}) + 2(y_2 + y_4 + \cdots \\ & + y_{2k-2})] + R_n, \end{aligned}$$

式中

$$R_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(\xi'') (a \leq \xi'' \leq b).$$

2531. 利用矩形公式($n = 12$), 近似地计算

$$\int_0^{2\pi} x \sin x dx$$

并把结果同精确答数比较.

解 $h = \frac{\pi}{6}.$

$$x_0 = 0, y_0 = 0;$$

$$x_1 = \frac{\pi}{6}, y_1 = \frac{\pi}{6} \sin \frac{\pi}{6} = 0.2618;$$

$$x_2 = \frac{\pi}{3}, y_2 = \frac{\pi}{3} \sin \frac{\pi}{3} = 0.9069;$$

$$x_3 = \frac{\pi}{2}, y_3 = \frac{\pi}{2} \sin \frac{\pi}{2} = 1.5708;$$

$$x_4 = \frac{2\pi}{3}, y_4 = \frac{2\pi}{3} \sin \frac{2\pi}{3} = 1.8138;$$

$$x_5 = \frac{5\pi}{6}, y_5 = \frac{5\pi}{6} \sin \frac{5\pi}{6} = 1.3090;$$

$$x_6 = \pi, y_6 = \pi \sin \pi = 0;$$

$$x_7 = \frac{7\pi}{6}, y_7 = \frac{7\pi}{6} \sin \frac{7\pi}{6} = -1.8326;$$

$$x_8 = \frac{4\pi}{3}, y_8 = \frac{4\pi}{3} \sin \frac{4\pi}{3} = -3.6276;$$

$$x_9 = \frac{3\pi}{2}, y_9 = \frac{3\pi}{2} \sin \frac{3\pi}{2} = -4.7124;$$

$$x_{10} = \frac{5\pi}{3}, y_{10} = \frac{5\pi}{3} \sin \frac{5\pi}{3} = -4.5345;$$

$$x_{11} = \frac{11\pi}{6}, y_{11} = \frac{11\pi}{6} \sin \frac{11\pi}{6} = -2.8798.$$

按矩形公式,得

$$\begin{aligned} & \int_0^{2\pi} x \sin x dx \\ &= \frac{\pi}{6} (y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \\ & \quad + y_8 + y_9 + y_{10} + y_{11}) \\ & \doteq -6.1390. \end{aligned}$$

实际上,

$$\begin{aligned} & \int_0^{2\pi} x \sin x dx \\ &= -x \cos x \Big|_0^{2\pi} + \int_0^{2\pi} \cos x dx \\ & \doteq -6.2832. \end{aligned}$$

利用梯形公式计算下列积分并估计它们的误差:

2532. $\int_0^1 \frac{dx}{1+x} \quad (n=8).$

解 $h = \frac{1}{8} = 0.125.$

$$\begin{aligned} x_0 &= 0, y_0 = 1; & \frac{y_0 + y_8}{2} &= 0.75, \\ x_8 &= 1, y_8 = 0.5; \end{aligned}$$

$$x_1 = \frac{1}{8} = 0.125, y_1 = 0.88889;$$

$$\begin{aligned}
x_2 &= 0.25, & y_2 &= 0.8; \\
x_3 &= 0.375, & y_3 &= 0.72727; \\
x_4 &= 0.5, & y_4 &= 0.66667; \\
x_5 &= 0.625, & y_5 &= 0.61538; \\
x_6 &= 0.75, & y_6 &= 0.57143; \\
x_7 &= 0.875, & y_7 &= 0.53333(+ \\
& & \sum_{i=1}^7 y_i &= 4.80297.
\end{aligned}$$

按梯形公式,得

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x} &= h \left(\frac{y_0 + y_8}{2} + \sum_{i=1}^7 y_i \right) \\
&= 0.125(0.75 + 4.80297) \\
&\doteq 0.69412,
\end{aligned}$$

误差为

$$|R_n| = \left| \frac{1}{12 \times 8^2} \cdot \frac{2}{(1+\xi)^3} \right| \quad (0 \leq \xi \leq 1).$$

于是,

$$|R_n| \leq \frac{2}{12 \times 8^2} < 0.0027 = 2.7 \times 10^{-3}$$

实际上,

$$\int_0^1 \frac{dx}{1+x} = \ln(1+x) \Big|_0^1 = \ln 2 \doteq 0.69315.$$

2533. $\int_0^1 \frac{dx}{1+x^3} \quad (n=12).$

解 $h = \frac{1}{12} = 0.08333.$

$$x_0 = 0, y_0 = 1;$$

$$x_{12} = 1, y_{12} = \frac{1}{2} = 0.5; \quad \frac{y_0 + y_{12}}{2} = 0.75,$$

$$x_1 = \frac{1}{12}, \quad y_1 = 0.99942;$$

$$x_2 = \frac{1}{6}, \quad y_2 = 0.99539;$$

$$x_3 = \frac{1}{4}, \quad y_3 = 0.98462;$$

$$x_4 = \frac{1}{3}, \quad y_4 = 0.96429;$$

$$x_5 = \frac{5}{12}, \quad y_5 = 0.93254;$$

$$x_6 = \frac{1}{2}, \quad y_6 = 0.88889;$$

$$x_7 = \frac{7}{12}, \quad y_7 = 0.83438;$$

$$x_8 = \frac{2}{3}, \quad y_8 = 0.77143;$$

$$x_9 = \frac{3}{4}, \quad y_9 = 0.70330;$$

$$x_{10} = \frac{5}{6}, \quad y_{10} = 0.63343;$$

$$x_{11} = \frac{11}{12}, \quad y_{11} = 0.56489(+$$

$$\sum_{i=1}^{11} y_i = 9.27258.$$

按梯形公式,得

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^3} &= h \left(\frac{y_0 + y_{12}}{2} + \sum_{i=1}^{11} y_i \right) \\ &= 0.0833(0.75 + 9.27258) \\ &\doteq 0.83518, \end{aligned}$$

误差为

$$|R_n| = \left| \frac{1}{12 \times 12^2} \cdot \frac{12\xi^4 - 6\xi}{(1 + \xi^3)^3} \right| \quad (0 \leq \xi \leq 1).$$

利用求极值的方法, 估计得 $\left| \frac{12\xi^4 - 6\xi}{(1 + \xi^3)^3} \right|$ 在 $[0, 1]$ 上不超过 2. 于是,

$$|R_n| \leq \frac{2}{12 \times 12^2} < 0.00116 = 1.16 \times 10^{-3}.$$

实际上,

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^3} &= \left[\frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} \right. \\ &\quad \left. + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} \right]_{x=0}^{x=1} \\ &= \frac{1}{3} \ln 2 + \frac{\pi}{3\sqrt{3}} \\ &= 0.83565. \end{aligned}$$

*) 利用 1881 题的结果。

$$2534. \quad \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{4} \sin^2 x} dx \quad (n=6).$$

$$\text{解} \quad h = \frac{\pi}{12} = 0.2618,$$

$$x_0 = 0, y_0 = 1;$$

$$x_6 = \frac{\pi}{2}, y_6 = 0.8660; \quad \frac{y_0 + y_6}{2} = 0.9330,$$

$$x_1 = \frac{\pi}{12}, y_1 = 0.9916;$$

$$x_2 = \frac{\pi}{6}, y_2 = 0.9682;$$

$$x_3 = \frac{\pi}{4}, y_3 = 0.9354;$$

$$x_4 = \frac{\pi}{3}, y_4 = 0.9014;$$

$$x_5 = \frac{5\pi}{12}, y_5 = 0.8756 \quad (+$$

$$\sum_{i=1}^5 y_i = 4.6722.$$

按梯形公式,得

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{4}\sin^2 x} dx &= h \left(\frac{y_0 + y_6}{2} + \sum_{i=1}^5 y_i \right) \\ &= 0.2618(0.9330 + 4.6724) \\ &\doteq 1.4674. \end{aligned}$$

误差为

$$|R_n| = \frac{\left(\frac{\pi}{2}\right)^3}{12 \times 6^2} |y''(\xi)|,$$

式中 $y = \sqrt{1 - \frac{1}{4}\sin^2 x}$, $0 \leq \xi \leq \frac{\pi}{2}$. 利用 $\frac{\sqrt{3}}{2} \leq y \leq 1$ 及 $y^2 = 1 - \frac{1}{4}\sin^2 x$, 依次求导可得 $|y''| \leq \frac{\sqrt{3}}{6}$. 于是,

$$|R_n| \leq \frac{\pi^3}{8 \times 12 \times 6^2} \cdot \frac{\sqrt{3}}{6} < 2.59 \times 10^{-3}.$$

利用辛普森公式计算下列积分:

2535. $\int_1^9 \sqrt{x} dx \quad (n=4).$

解 $h=2.$

$$x_0 = 1, y_0 = 1;$$

$$x_1 = 3, y_1 = \sqrt{3} = 1.732;$$

$$x_2 = 5, y_2 = \sqrt{5} = 2.236;$$

$$x_3 = 7, y_3 = \sqrt{7} = 2.646;$$

$$x_4 = 9, y_4 = 3;$$

按辛普森公式,得

$$\begin{aligned}\int_1^9 \sqrt{x} dx &= \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2] \\ &= \frac{2}{3} [4 + 4(1.732 + 2.646) \\ &\quad + 2(2.236)] \\ &\doteq 17.323.\end{aligned}$$

实际上,

$$\int_1^9 \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_1^9 = \frac{52}{3} \doteq 17.333.$$

$$2536. \int_0^{\pi} \sqrt{3 + \cos x} dx \quad (n = 6).$$

解 $h = \frac{\pi}{6}.$

$$x_0 = 0, y_0 = 2;$$

$$\begin{aligned}x_1 &= \frac{\pi}{6}, y_1 = \sqrt{3 + \cos \frac{\pi}{6}} = \sqrt{3.866} \\ &= 1.966;\end{aligned}$$

$$x_2 = \frac{\pi}{3}, y_2 = \sqrt{3 + \cos \frac{\pi}{3}} = \sqrt{3.5} = 1.871;$$

$$x_3 = \frac{\pi}{2}, y_3 = \sqrt{3 + \cos \frac{\pi}{2}} = \sqrt{3} = 1.732;$$

$$x_4 = \frac{2\pi}{3}, y_4 = \sqrt{3 + \cos \frac{2\pi}{3}} = \sqrt{2.5} = 1.581;$$

$$x_5 = \frac{5\pi}{6}, \quad y_5 = \sqrt{3 + \cos \frac{5\pi}{6}} = \sqrt{2.134} \\ = 1.461;$$

$$x_6 = \pi, \quad y_6 = \sqrt{3 + \cos \pi} = \sqrt{2} = 1.414.$$

按辛普森公式,得

$$\int_0^{\pi} \sqrt{3 + \cos x} dx \\ = \frac{\pi}{18} [(2 + 1.414) + 4(1.966 + 1.736 + 1.461) \\ + 2(1.871 + 1.581)] \\ = 5.4053.$$

2537⁺ . $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx \quad (n = 10).$

解 $h = \frac{\pi}{20}.$

$$x_0 = 0, \quad y_0 = 1;$$

$$x_1 = \frac{\pi}{20}, \quad y_1 = \frac{20}{\pi} \sin \frac{\pi}{20} = 0.99589;$$

$$x_2 = \frac{\pi}{10}, \quad y_2 = \frac{10}{\pi} \sin \frac{\pi}{10} = 0.98363;$$

$$x_3 = \frac{3\pi}{20}, \quad y_3 = \frac{20}{3\pi} \sin \frac{3\pi}{20} = 0.96340;$$

$$x_4 = \frac{\pi}{5}, \quad y_4 = \frac{5}{\pi} \sin \frac{\pi}{5} = 0.93549;$$

$$x_5 = \frac{\pi}{4}, \quad y_5 = \frac{4}{\pi} \sin \frac{\pi}{4} = 0.90032;$$

$$x_6 = \frac{3\pi}{10}, \quad y_6 = \frac{10}{3\pi} \sin \frac{3\pi}{10} = 0.85839;$$

$$x_7 = \frac{7\pi}{20}, \quad y_7 = \frac{20}{7\pi} \sin \frac{7\pi}{20} = 0.81033;$$

$$x_5 = \frac{5}{6}, y_5 = 1.3748;$$

$$x_6 = 1, y_6 = 1.4427.$$

按辛普森公式,得

$$\begin{aligned} \int_0^1 \frac{x dx}{\ln(1+x)} &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) \\ &\quad + 2(y_2 + y_4)] \\ &= \frac{1}{18} [(1 + 1.4427) + 4(1.0812 \\ &\quad + 1.2332 + 1.3748) + 2(1.1587 \\ &\quad + 1.3051)] \\ &\doteq 1.2293. \end{aligned}$$

2539. 取 $n = 10$, 计算加达郎常数

$$G = \int_0^1 \frac{\arctg x}{x} dx.$$

解 $h = \frac{1}{10}.$

$$x_0 = 0, y_0 = 1;$$

$$x_1 = 0.1, y_1 = 0.99669;$$

$$x_2 = 0.2, y_2 = 0.98698;$$

$$x_3 = 0.3, y_3 = 0.97152;$$

$$x_4 = 0.4, y_4 = 0.95127;$$

$$x_5 = 0.5, y_5 = 0.92730;$$

$$x_6 = 0.6, y_6 = 0.90070;$$

$$x_7 = 0.7, y_7 = 0.87247;$$

$$x_8 = 0.8, y_8 = 0.84343;$$

$$x_9 = 0.9, y_9 = 0.81424;$$

$$x_{10} = 1, \quad y_{10} = 0.78540.$$

按辛普森公式,得

$$\begin{aligned} G &= \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) \\ &\quad + 2(y_2 + y_4 + y_6 + y_8)] \\ &= \frac{1}{30} (1.78540 + 18.32888 + 7.36476) \\ &= 0.91597. \end{aligned}$$

2540. 利用公式

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

计算数 π 精确到 10^{-5} .

解 利用辛普森公式计算其误差

$$R_n(x) = -\frac{(b-a)^5}{180n^4} f^{(4)}(\xi) \quad (a \leq \xi \leq b).$$

现在 $f(x) = \frac{1}{1+x^2}$, 事实上, 它是 $y = \arctan x$ 的导数, 因而

$$f^{(4)}(x) = (\arctan x)^{(5)}.$$

利用第二章 1218 题的结果得知

$$f^{(4)}(x) = \frac{24}{(1+x^2)^{\frac{5}{2}}} \sin\left(5 \arctan \frac{1}{x}\right).$$

在区间 $[0, 1]$ 上,

$$|f^{(4)}(x)| \leq 24,$$

所以

$$|R_n(x)| \leq \frac{24}{180n^4}.$$

欲误差小于 0.00001, 只需

$$\frac{24}{180n^4} < \frac{1}{100000},$$

即只需取 $n = 12$, 就有 $|R_n| \leqslant 6.5 \times 10^{-6}$.

其次, 我们还必须加进近似于函数值的误差, 设法使这个新的误差小于 3.6×10^{-6} , 这样, 就能保证总误差小于 10^{-5} . 为了这个目的, 只要计算 $\frac{1}{1+x^2}$ 的值到六位小数精确到 0.5×10^{-6} 就够了.

现取 $n = 12$, 则有

$$x_0 = 0, y_0 = 1;$$

$$x_1 = \frac{1}{12}, y_1 = 0.993103;$$

$$x_2 = \frac{1}{6}, y_2 = 0.972973;$$

$$x_3 = \frac{1}{4}, y_3 = 0.941176;$$

$$x_4 = \frac{1}{3}, y_4 = 0.900000;$$

$$x_5 = \frac{5}{12}, y_5 = 0.852071;$$

$$x_6 = \frac{1}{2}, y_6 = 0.800000;$$

$$x_7 = \frac{7}{12}, y_7 = 0.746114;$$

$$x_8 = \frac{2}{3}, y_8 = 0.692308;$$

$$x_9 = \frac{3}{4}, y_9 = 0.640000;$$

$$x_{10} = \frac{5}{6}, y_{10} = 0.590164;$$

$$x_{11} = \frac{11}{12}, y_{11} = 0.543396;$$

$$x_{12} = 1, y_{12} = 0.500000.$$

最后得到

$$\begin{aligned} \frac{\pi}{4} &= \int_0^1 \frac{dx}{1+x^2} \\ &= \frac{1}{36} [(y_0 + y_{12}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 \\ &\quad + y_{11}) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})] \\ &= 0.785398, \end{aligned}$$

所以

$$\pi = 0.785398 \times 4 = 3.14159,$$

精确到 0.00001.

2541. 计算

$$\int_0^1 e^{x^2} dx$$

精确到 0.001.

解 采用辛普森公式计算, 则其误差

$$\begin{aligned} R_n(x) &= -\frac{1}{180n^4} 2e^{\xi^2} (8\xi^4 + 24\xi^2 + 6) \\ &\quad (0 < \xi < 1), \end{aligned}$$

$$\text{故有 } |R_n(x)| < \frac{1}{180n^4} \cdot 2e \cdot 38.$$

要 $|R_n(x)| < 10^{-3}$, 只须 $\frac{2 \cdot 38 \cdot e^1}{180n^4} < 10^{-3}$, 即只须取

$$n = 6.$$

现取 $n = 6$, 则有

$$x_0 = 0, y_0 = 1;$$

$$x_1 = \frac{1}{6}, y_1 = e^{\frac{1}{36}} = 1.0282;$$

$$x_2 = \frac{1}{3}, y_2 = e^{\frac{1}{9}} = 1.1175;$$

$$x_3 = \frac{1}{2}, y_3 = e^{\frac{1}{4}} = 1.2840;$$

$$x_4 = \frac{2}{3}, y_4 = e^{\frac{4}{9}} = 1.5596;$$

$$x_5 = \frac{5}{6}, y_5 = e^{\frac{25}{36}} = 2.0026;$$

$$x_6 = 1, y_6 = e = 2.7183.$$

于是,

$$\begin{aligned} \int_0^1 e^{x^2} dx &= \frac{1}{18} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) \\ &\quad + 2(y_2 + y_4)] \doteq 1.463. \end{aligned}$$

2542. 计算

$$\int_0^1 (e^x - 1) \ln \frac{1}{x} dx \text{ 精确到 } 10^{-4}.$$

解 对于函数 $f(x) = e^x$ 在 $0 \leq x \leq 1$ 上采用台劳展式以及相应的拉格朗日余项公式来估算误差:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \Delta_{n+1},$$

其中

$$\begin{aligned} \Delta_{n+1} &= \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} \\ &= \frac{e^{\theta x}}{(n+1)!} x^{n+1} \quad (0 < \theta < 1). \end{aligned}$$

于是

$$|\Delta_{n+1}| \leq \frac{e}{(n+1)!} x^{n+1},$$

从而原来的积分数值为

$$\begin{aligned} I &= \int_0^1 (e^x - 1) \ln \frac{1}{x} dx \\ &= \sum_{k=1}^n \frac{1}{k!} \int_0^1 x^k \ln \frac{1}{x} dx + R_{n+1}, \end{aligned}$$

其中

$$\begin{aligned} |R_{n+1}| &= \left| \int_0^1 \Delta_{n+1} \ln \frac{1}{x} dx \right| \\ &\leq \frac{e}{(n+1)!} \int_0^1 x^{n+1} \ln \frac{1}{x} dx. \end{aligned}$$

记 $I_k = \int_0^1 x^k \ln \frac{1}{x} dx$ ($k \geq 1$), 则有

$$\begin{aligned} I_k &= \frac{1}{k+1} \int_0^1 \ln \frac{1}{x} d(x^{k+1}) \\ &= \frac{1}{k+1} x^{k+1} \ln \frac{1}{x} \Big|_0^1 + \frac{1}{k+1} \int_0^1 x^k dx \\ &= \frac{1}{(k+1)^2}. \end{aligned}$$

如果取 $n = 5$, 则有

$$\begin{aligned} |R_6| &\leq \frac{e}{6!} I_6 = \frac{e}{6!} \cdot \frac{1}{7^2} = \frac{e}{7 \times 7!} \\ &= \frac{e}{35280} < \frac{3}{35280} < \frac{1}{1.1 \times 10^4} < 10^{-4}. \end{aligned}$$

记 $I = J + R_6$, 则有

$$\begin{aligned} J &= \sum_{k=1}^5 \frac{1}{k!} I_k = \sum_{k=1}^5 \frac{1}{k!} \cdot \frac{1}{(k+1)^2} \\ &= \sum_{k=1}^5 \frac{1}{(k+1)! (k+1)} \\ &= \frac{1}{2!2} + \frac{1}{3!3} + \frac{1}{4!4} + \frac{1}{5!5} + \frac{1}{6!6} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} + \frac{1}{18} + \frac{1}{96} + \frac{1}{600} + \frac{1}{4320} \\
&= 0.31787^+ = 0.3179 + \Delta',
\end{aligned}$$

其中 $|\Delta'| \leq 0.00004 = 4 \times 10^{-5}$ 且 $\Delta' < 0$.

注意到由 $\Delta_{n+1} > 0$ 即可推知 $R_{n+1} > 0$. 于是

$$\begin{aligned}
I &= J + R_6 = 0.3179 + (R_6 + \Delta') \\
&= 0.3179 + (R_6 - |\Delta'|) = 0.3179 + \Delta,
\end{aligned}$$

且有 $I \doteq 0.3179$, 而此时其相应的误差已有

$$\begin{aligned}
|\Delta| = |R_6 - |\Delta'|| &\leq \begin{cases} R_6, & \text{若 } |\Delta'| \leq R_6, \\ |\Delta'|, & \text{若 } |\Delta'| > R_6 \end{cases} \\
&\leq \max(R_6, |\Delta'|) < 10^{-4}.
\end{aligned}$$

注 本题不能直接利用辛普森公式来计算所给的定积分的近似值, 因为被积函数 $(e^x - 1)\ln \frac{1}{x}$ 的四阶导函数在 $x = 0$ 的右近旁是无界的, 从而不能估计出误差. 所以, 上面我们用台劳公式来作近似计算. 这样, 计算以及估计误差都较为简单. 当然, 也可间接地利用辛普森公式来计算所给定积分的近似值, 这时需要或者改变被积函数或者把积分区间分成两个. 例如, 我们可以改变被积函数如下: 令

$$I = \int_0^1 (e^x - 1) \ln \frac{1}{x} dx = - \int_0^1 (e^x - 1) \ln x dx,$$

设 $f(x) = (e^x - 1)\ln x$, 若补充定义

$$f(0) = \lim_{x \rightarrow +0} f(x) = 0,$$

则 $f(x)$ 是 $0 \leq x \leq 1$ 上的连续函数. 由于

$$f'(x) = e^x \ln x + \frac{e^x - 1}{x}$$

$$= f(x) + \frac{e^x - 1}{x} + \ln x (0 < x \leq 1),$$

故

$$\begin{aligned} \int_0^1 f'(x) dx &= \int_0^1 f(x) dx + \int_0^1 \frac{e^x - 1}{x} dx \\ &\quad + \int_0^1 \ln x dx. \end{aligned}$$

注意到

$$\begin{aligned} \int_0^1 f'(x) dx &= f(1) - f(0) = 0, \\ \int_0^1 \ln x dx &= (x \ln x - x) \Big|_0^1 = -1, \end{aligned}$$

得

$$I = \int_0^1 \frac{e^x - 1}{x} dx = 1.$$

于是, 我们把求 $\int_0^1 (e^x - 1) \ln \frac{1}{x} dx$ 的近似值问题, 归结为求 $\int_0^1 \frac{e^x - 1}{x} dx$ 的近似值问题. 令 $g(x) = \frac{e^x - 1}{x}$, 并补充定义

$$g(0) = \lim_{x \rightarrow +0} g(x) = 1,$$

则 $g(x)$ 是 $0 \leq x \leq 1$ 上的连续函数. 由求高阶导数的莱布尼兹法则, 易得

$$g^{(n)}(x) = \frac{e^x P_n(x) - (-1)^n n!}{x^{n+1}} (0 < x \leq 1),$$

$$\text{其中 } P_n(x) = \sum_{k=0}^n C_n^k (-1)^k k! x^{n-k} \quad (n = 1, 2, \dots).$$

下面证明 $g^{(n)}(0)$ 存在并且 $g^{(n)}(0) = \frac{1}{n+1}$ ($n = 1, 2, \dots$). 首先, 由洛比塔法则, 我们有

$$\begin{aligned}
\lim_{x \rightarrow +0} g^{(n)}(x) &= \lim_{x \rightarrow +0} \frac{e^x P_n(x) - (-1)^n n!}{x^{n+1}} \\
&= \lim_{x \rightarrow +0} \frac{e^x [P_n(x) + P'_n(x)]}{(n+1)x^n} \\
&= \lim_{x \rightarrow +0} \frac{e^x x^n}{(n+1)x^n} = \frac{1}{n+1} \quad (n = 1, 2, \dots).
\end{aligned}$$

于是,根据中值定理,得

$$\begin{aligned}
g'(0) &= \lim_{x \rightarrow +0} \frac{g(x) - g(0)}{x - 0} = \lim_{\xi \rightarrow +0} g'(\xi) \\
&= \frac{1}{2} \quad (0 < \xi < x),
\end{aligned}$$

今假定 $g^{(n)}(0)$ 存在且 $g^{(n)}(0) = \frac{1}{n+1}$, 于是,

$$\begin{aligned}
g^{(n+1)}(0) &= \lim_{x \rightarrow +0} \frac{g^{(n)}(x) - g^{(n)}(0)}{x - 0} \\
&= \lim_{x \rightarrow +0} g^{(n+1)}(\eta) = \frac{1}{n+2} \quad (0 < \eta < x).
\end{aligned}$$

根据数学归纳法,知 $g^{(n)}(0)$ 存在且

$$g^{(n)}(0) = \frac{1}{n+1} \quad (n = 1, 2, \dots).$$

由此又知 $g^{(n)}(x)$ 是 $0 \leq x \leq 1$ 上的连续函数 ($n = 1, 2, \dots$). 令 $h(x) = e^x P_n(x) - (-1)^n n!$. 由于

$$\begin{aligned}
h'(x) &= e^x [P_n(x) + P'_n(x)] = e^x x^n > 0 \\
&\quad (\text{当 } 0 < x \leq 1 \text{ 时}),
\end{aligned}$$

故 $h(x)$ 在 $[0, 1]$ 上是严格增大的,从而

$$h(x) > h(0) = 0 \quad (\text{当 } 0 < x \leq 1 \text{ 时}).$$

因此,当 $0 < x \leq 1$ 时 $g^{(n)}(x) > 0$ ($n = 1, 2, \dots$), 所以 $g^{(n-1)}(x)$ 是 $0 \leq x \leq 1$ 上的严格增函数 ($n = 1, 2, \dots$). 特别, $g^{(4)}(x)$ 当然是 $0 \leq x \leq 1$ 上的严格增函数. 于

令 $u = \frac{1-x}{x}$ ($0 < x < 1$), 则 $\frac{1}{x} = 1 + u$ ($u > 0$).

于是, 当 $0 < x < 1$ 时, 有

$$\begin{aligned} 0 &< (e^x - 1) \ln \frac{1}{x} = (e^x - 1) \ln(1 + u) \\ &< (e^x - 1)u = \frac{1-x}{x}(e^x - 1) < \frac{e^x - 1}{x}. \end{aligned}$$

前面已证函数 $g(x) = \frac{e^x - 1}{x}$ 在 $0 \leq x \leq 1$ 上是严格增大的 (注意, 规定 $g(0) = \lim_{x \rightarrow +0} \frac{e^x - 1}{x} = 1$), 故当 $0 < x < 1$ 时, 有

$$1 < \frac{e^x - 1}{x} < g(1) = e - 1 < 2;$$

从而

$$\begin{aligned} 0 &< \int_0^{10^{-5}} (e^x - 1) \ln \frac{1}{x} dx < \int_0^{10^{-5}} \frac{e^x - 1}{x} dx \\ &< 2 \int_0^{10^{-5}} dx = 0.2 \times 10^{-4}. \end{aligned}$$

求出函数 $(e^x - 1) \ln \frac{1}{x}$ 的四阶导函数的表达式后, 易知它在闭区间 $10^{-5} \leq x \leq 1$ 上是连续的, 从而是有界的, 并且不难估计出其绝对值的上界. 因此, 可利用辛普森公式计算积分

$$\int_{10^{-5}}^1 (e^x - 1) \ln \frac{1}{x} dx$$

的近似值, 使误差的绝对值小于 0.8×10^{-4} . 显然, 若以此作为积分 $\int_0^1 (e^x - 1) \ln \frac{1}{x}$ 的近似值, 则其误差的

绝对值小于 10^{-4} . 由于计算较繁, 从略.

2543. 近似地计算概率积分

$$\int_0^{+\infty} e^{-x^2} dx.$$

解 作变换

$$x = \frac{t}{1-t},$$

则积分

$$\int_0^{+\infty} e^{-x^2} dx = \int_0^1 e^{-(\frac{t}{1-t})^2} \frac{1}{(1-t)^2} dt.$$

由于题中对精确度未提出明确要求, 故 n 可任取. 例如

取 $n = 2k = 18, \Delta t = \frac{1}{18}$, 则有

$$t_0 = 0, y_0 = 1;$$

$$t_1 = \frac{1}{18}, 4y_1 = 4.46894;$$

$$t_2 = \frac{1}{9}, 2y_2 = 2.49201;$$

$$t_3 = \frac{1}{6}, 4y_3 = 5.53415;$$

$$t_4 = \frac{2}{9}, 2y_4 = 3.04696;$$

$$t_5 = \frac{5}{18}, 4y_5 = 6.61414;$$

$$t_6 = \frac{1}{3}, 2y_6 = 3.50460;$$

$$t_7 = \frac{7}{18}, 4y_7 = 7.14411;$$

$$t_8 = \frac{4}{9}, 2y_8 = 3.41685;$$

$$t_9 = \frac{1}{2}, 4y_9 = 5.88607;$$

$$t_{10} = \frac{5}{9}, 2y_{10} = 2.12232;$$

$$t_{11} = \frac{11}{18}, 4y_{11} = 2.23855;$$

$$t_{12} = \frac{2}{3}, 2y_{12} = 0.32968;$$

$$t_{13} = \frac{13}{18}, 4y_{13} = 0.06009;$$

$$t_{14} = \frac{7}{9}, 2y_{14} = 0.00010;$$

$$t_{15} = \frac{5}{6}, 4y_{15} = 0;$$

$$t_{16} = \frac{8}{9}, 2y_{16} = 0;$$

$$t_{17} = \frac{17}{18}, 4y_{17} = 0;$$

$$t_{18} = 1, y_{18} = \lim_{t \rightarrow 1} e^{-\left(\frac{t}{1-t}\right)^2} \left(\frac{1}{1-t}\right)^2 = 0.$$

按辛普森公式,得

$$\begin{aligned} \int_0^{+\infty} e^{-x^2} dx &= \int_0^1 e^{-\left(\frac{t}{1-t}\right)^2} \frac{1}{(1-t)^2} dt \\ &= \frac{1}{54} (1 + 4.46894 + 2.49201 \\ &\quad + 5.53415 + 3.04696 + 6.61414 \\ &\quad + 3.50460 + 7.14411 + 3.41685 \\ &\quad + 3.88607 + 2.12232 + 2.23855 \\ &\quad + 0.32968 + 0.06009 + 0.00010) \\ &= \frac{47.85857}{54} \doteq 0.88627. \end{aligned}$$

2544. 近似地求出半轴为 $a = 10$ 及 $b = 6$ 的椭圆的周长.

解 设椭圆的参数方程为

$$x = 10\cos t, y = 6\sin t.$$

于是有 $ds = \sqrt{x_i'^2 + y_i'^2} dt = 10 \sqrt{1 - \frac{16}{25}\sin^2 t} dt$,

从而得椭圆的周长为

$$s = 4 \int_0^{\frac{\pi}{2}} ds = 40 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25}\sin^2 t} dt.$$

现取 $n = 2k = 6$ 近似计算积分

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25}\sin^2 t} dt.$$

注意到 $\sin^2 \frac{\pi}{12} = \frac{2 - \sqrt{3}}{4}, \sin^2 \frac{5\pi}{12} = \frac{2 + \sqrt{3}}{4}$,

$h = \frac{\pi}{12}$, 即有

$$t_0 = 0, y_0 = 1;$$

$$\begin{aligned} t_1 &= \frac{\pi}{12}, 4y_1 = 4 \sqrt{1 - \frac{16}{25} \cdot \frac{1}{4} (2 - \sqrt{3})} \\ &= 3.913; \end{aligned}$$

$$t_2 = \frac{\pi}{6}, 2y_2 = 2 \sqrt{1 - \frac{16}{25} \cdot \frac{1}{4}} = 1.833;$$

$$t_3 = \frac{\pi}{4}, 4y_3 = 4 \sqrt{1 - \frac{16}{25} \cdot \frac{1}{2}} = 3.293;$$

$$t_4 = \frac{\pi}{3}, 2y_4 = 2 \sqrt{1 - \frac{16}{25} \cdot \frac{3}{4}} = 1.442;$$

$$t_5 = \frac{5\pi}{12}, 4y_5 = 4 \sqrt{1 - \frac{16}{25} \cdot \frac{1}{4} (2 + \sqrt{3})}$$

$$= 2.539;$$

$$t_6 = \frac{\pi}{2}, y_6 = \sqrt{1 - \frac{16}{25}} = 0.6.$$

按辛普森公式,得

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25} \sin^2 t} dt \\ &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{\pi}{36} (1 + 0.6 + 3.913 + 3.298 + 2.539 + 1.833 \\ & \quad + 1.442) \\ &\doteq 1.276, \end{aligned}$$

所以,椭圆周长的近似值为

$$\begin{aligned} s &= 40 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25} \sin^2 t} dt \\ &= 40 \times 1.276 = 51.04. \end{aligned}$$

2545. 取 $\Delta x = \frac{\pi}{3}$, 按点子作出函数

$$y = \int_0^x \frac{\sin t}{t} dt \quad (0 \leq x \leq 2\pi)$$

的图形.

解 取 $n = 2k = 6$ 计算函数 $y = \int_0^x \frac{\sin t}{t} dt$ 的值.

先计算 $y = \int_0^{\frac{\pi}{3}} \frac{\sin t}{t} dt$. 由于 $h = \frac{\pi}{18}$, 且

$$t_0 = 0, y_0 = 1;$$

$$t_1 = \frac{\pi}{18}, 4y_1 = 3.980;$$

$$t_2 = \frac{\pi}{9}, 2y_2 = 1.960;$$

$$t_3 = \frac{\pi}{6}, 4y_3 = 3.820;$$

$$t_4 = \frac{2\pi}{9}, 2y_4 = 1.841;$$

$$t_5 = \frac{5\pi}{18}, 4y_5 = 3.511;$$

$$t_6 = \frac{\pi}{3}, y_6 = 0.827.$$

按辛普森公式,得

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{54} (1 + 0.827 + 3.980 + 3.820 \\ &\quad + 3.511 + 1.960 + 1.841) \\ &\doteq 0.99. \end{aligned}$$

再计算 $y = \int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt$. 由于 $h = \frac{\pi}{9}$, 且

$$t_0 = 0, y_0 = 1;$$

$$t_1 = \frac{\pi}{9}, 4y_1 = 3.919;$$

$$t_2 = \frac{2\pi}{9}, 2y_2 = 1.841;$$

$$t_3 = \frac{\pi}{3}, 4y_3 = 3.308;$$

$$t_4 = \frac{4\pi}{9}, 2y_4 = 1.411;$$

$$t_5 = \frac{5\pi}{9}, 4y_5 = 2.257;$$

$$t_6 = \frac{2\pi}{3}, y_6 = 0.413.$$

所以,

$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{27} (1 + 0.413 + 3.919 + 3.308 \\ &\quad + 2.257 + 1.841 + 1.411) \\ &\doteq 1.65.\end{aligned}$$

选取适当的 n , 类似地可求得

$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

列表作图如下(图 4.55):

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

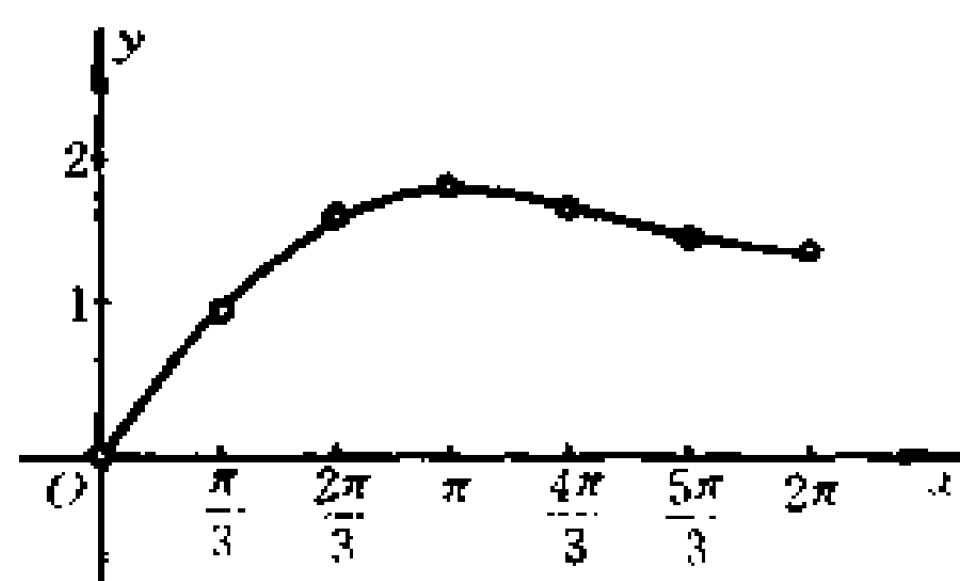


图 4.55

$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{27} (1 + 0.413 + 3.919 + 3.308 \\ &\quad + 2.257 + 1.841 + 1.411) \\ &\doteq 1.65.\end{aligned}$$

选取适当的 n , 类似地可求得

$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

列表作图如下(图 4.55):

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

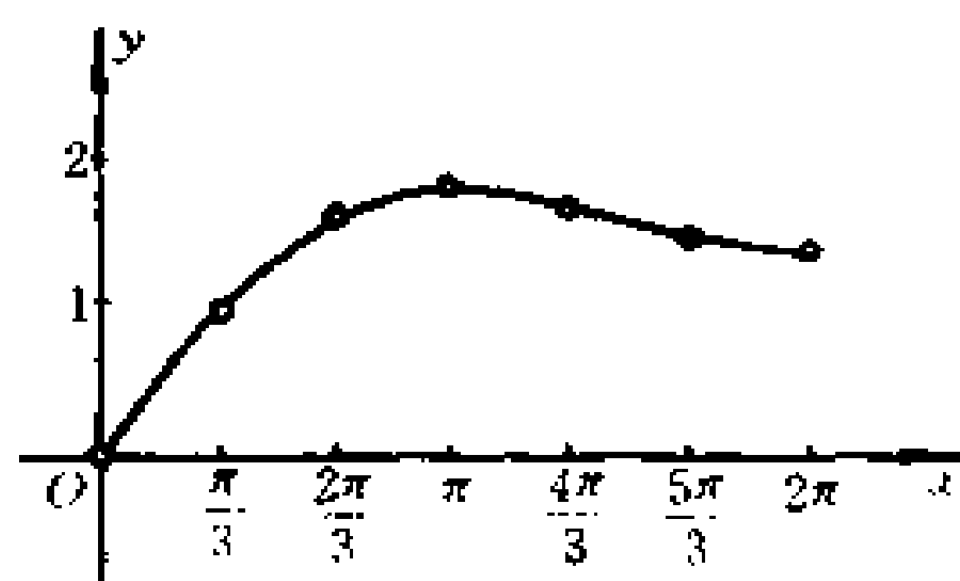


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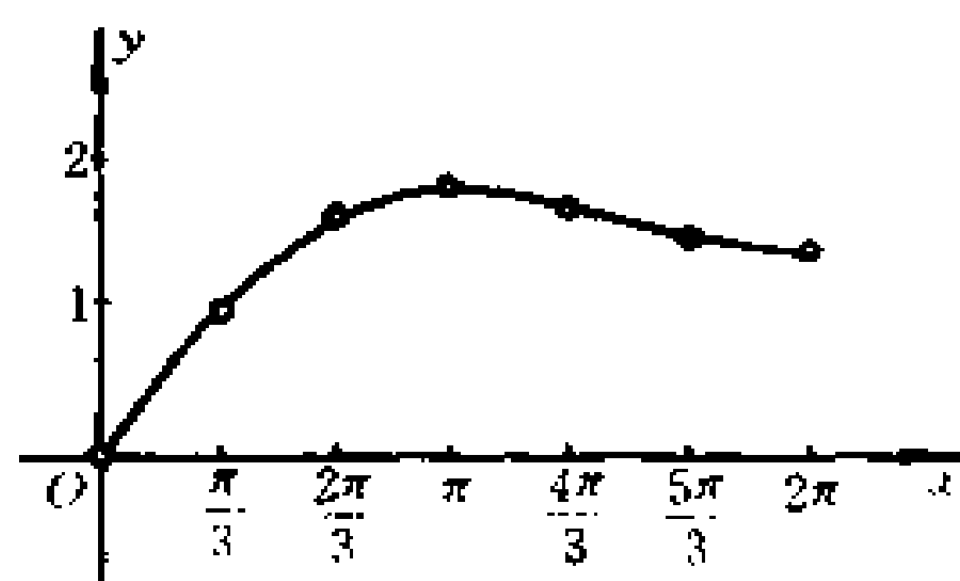


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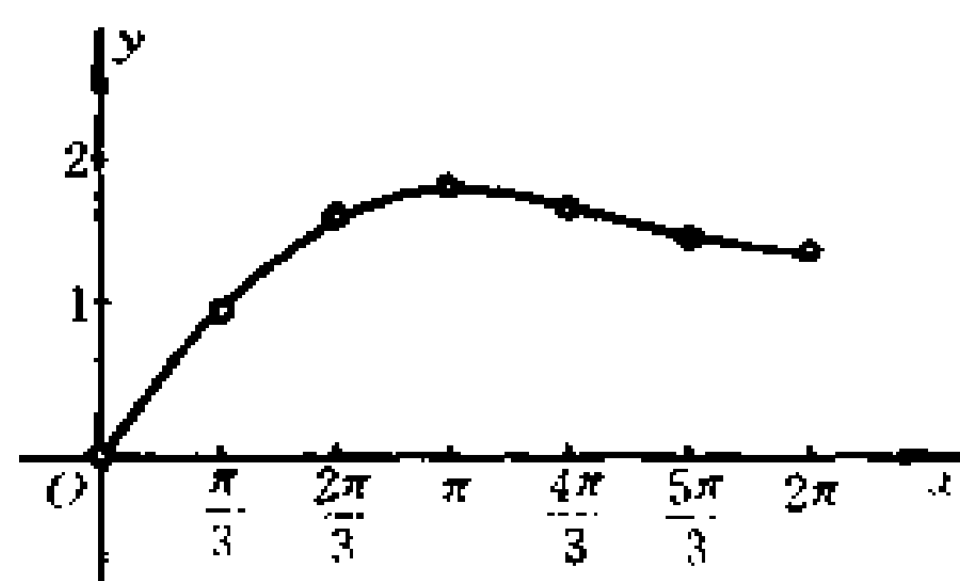


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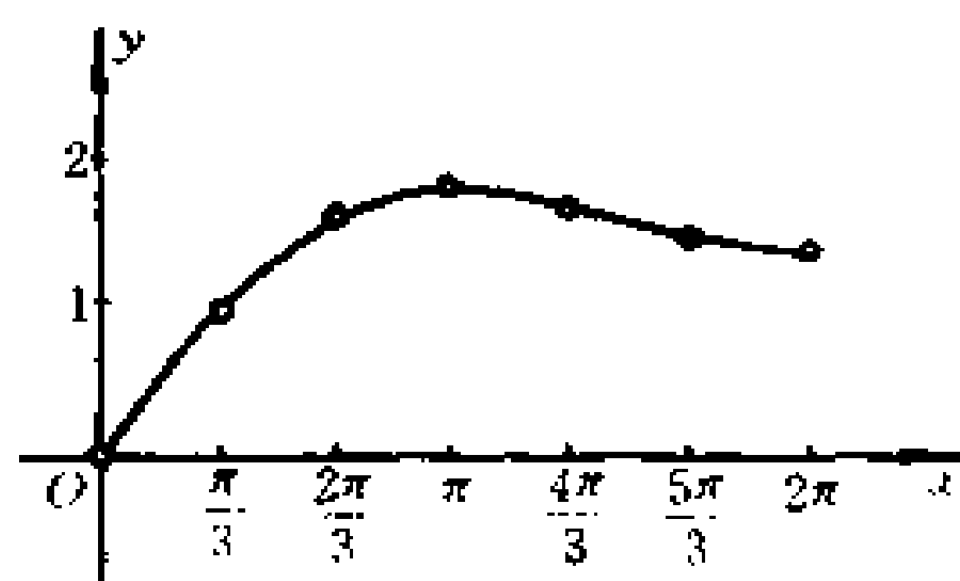


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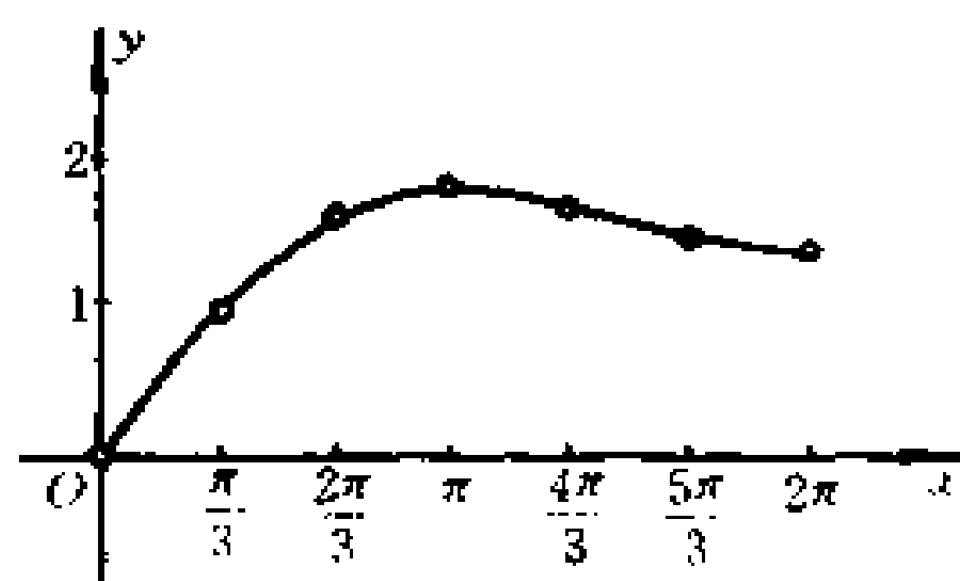


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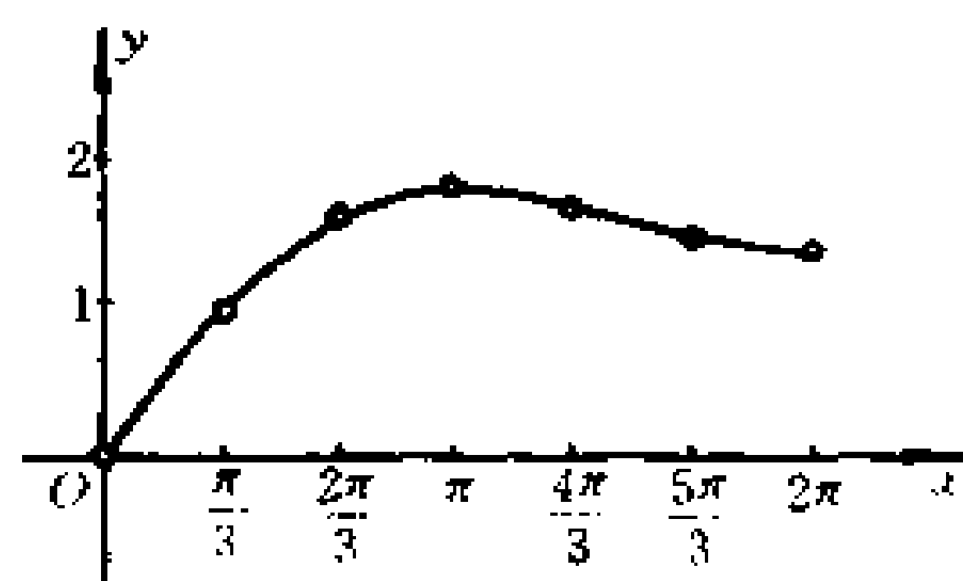


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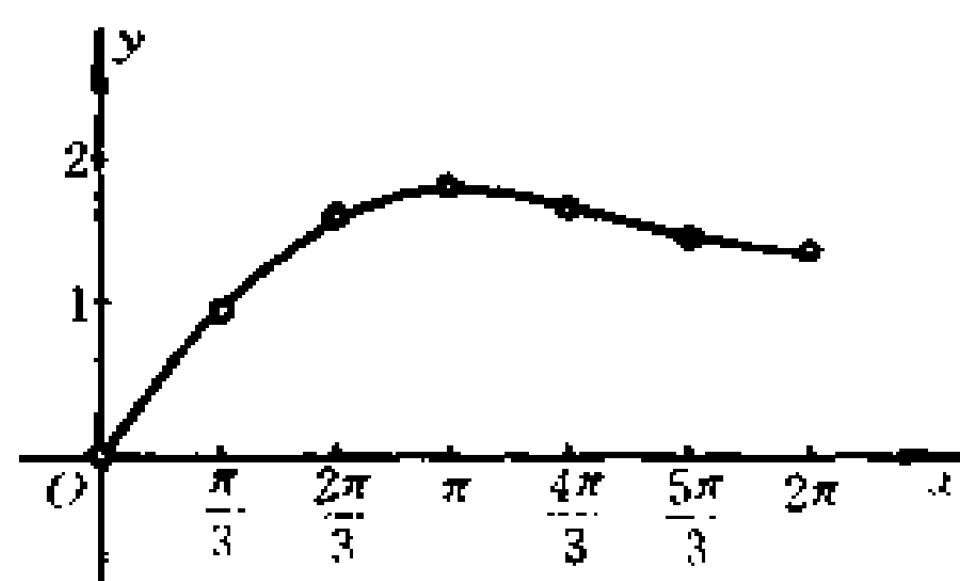


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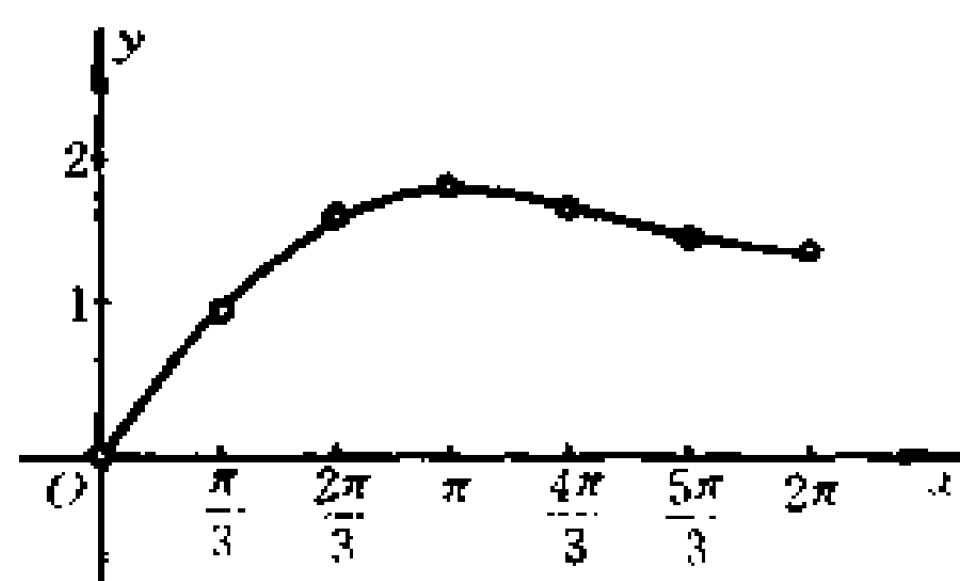


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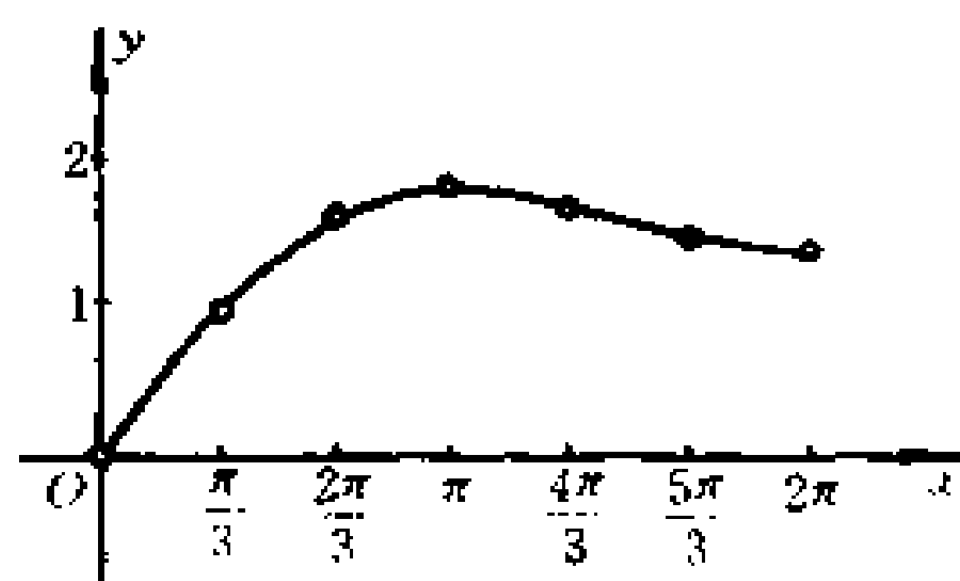


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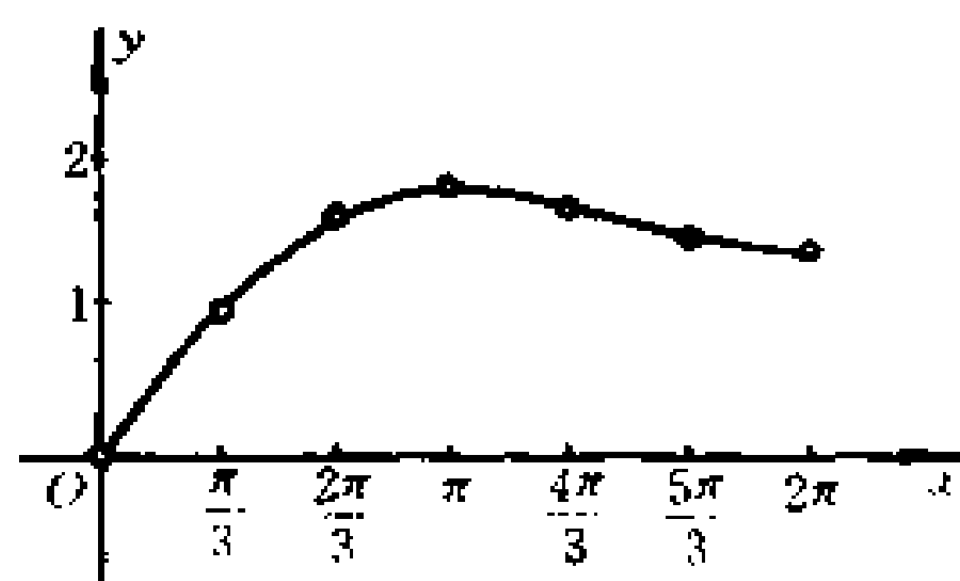


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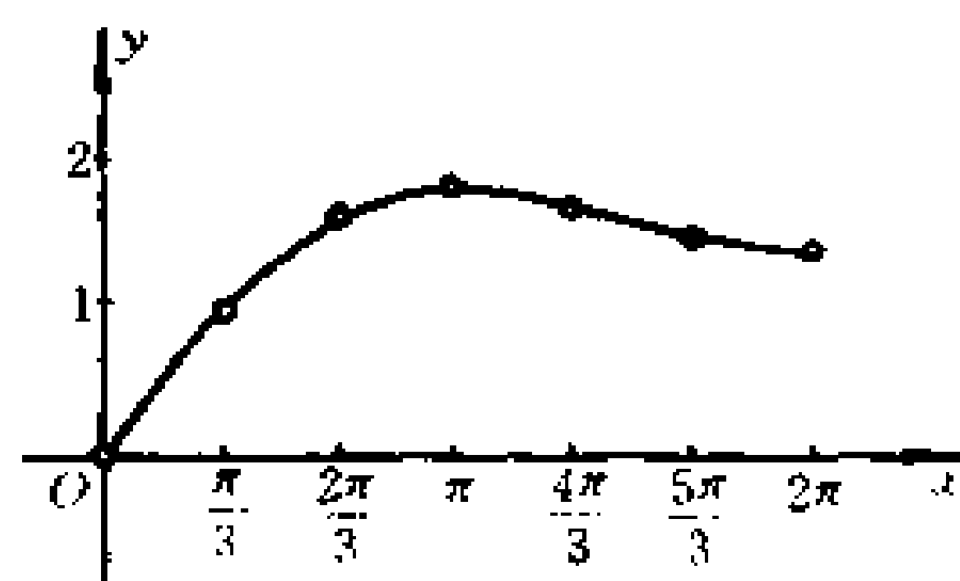


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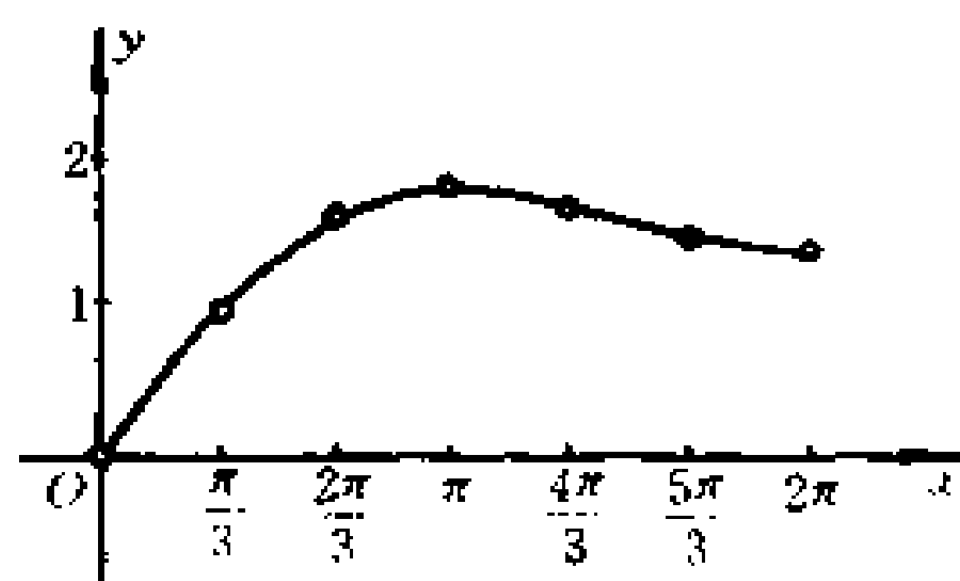


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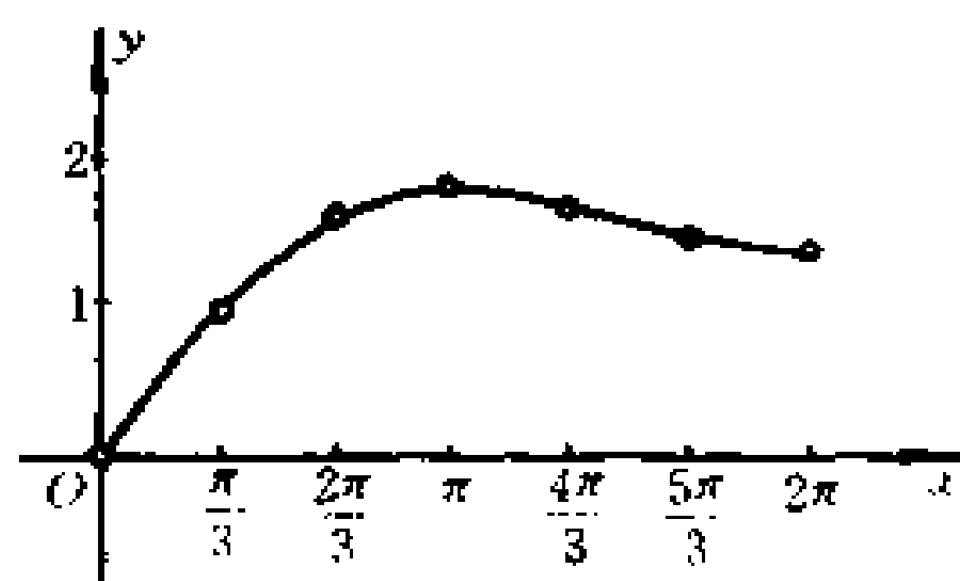


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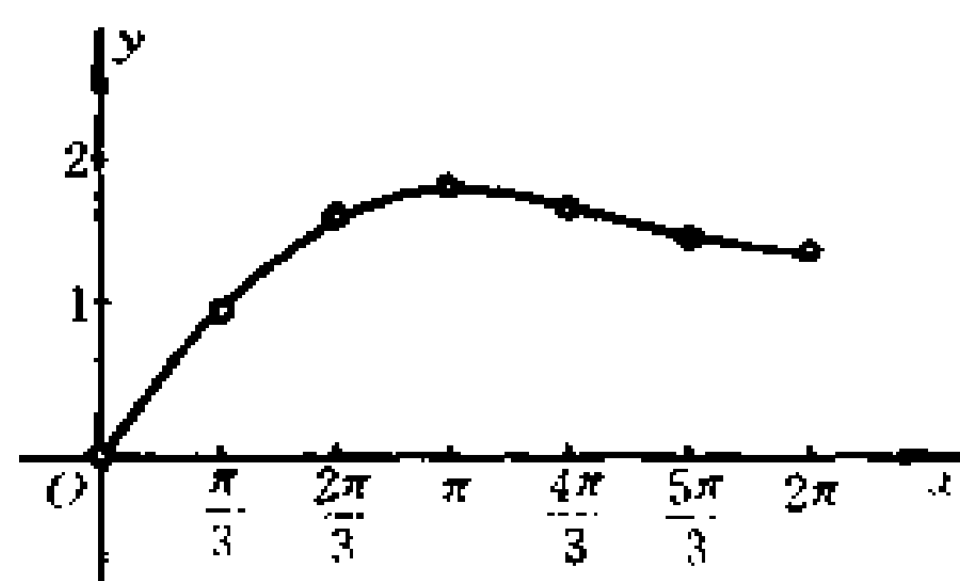


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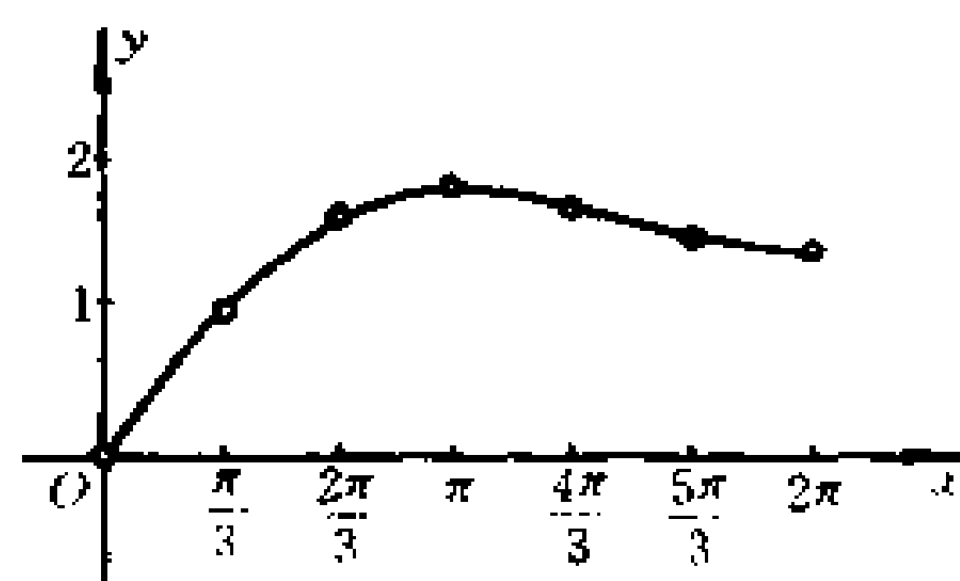


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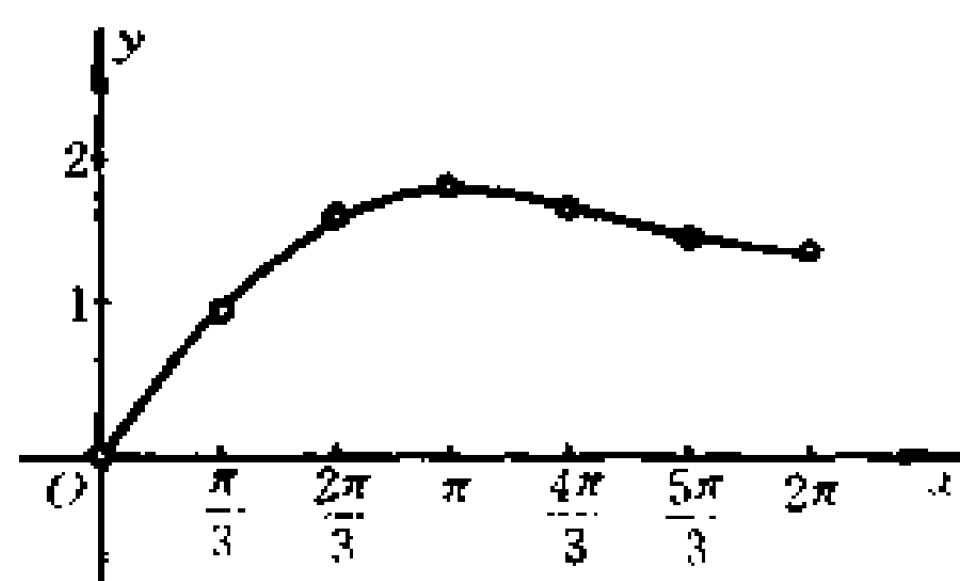


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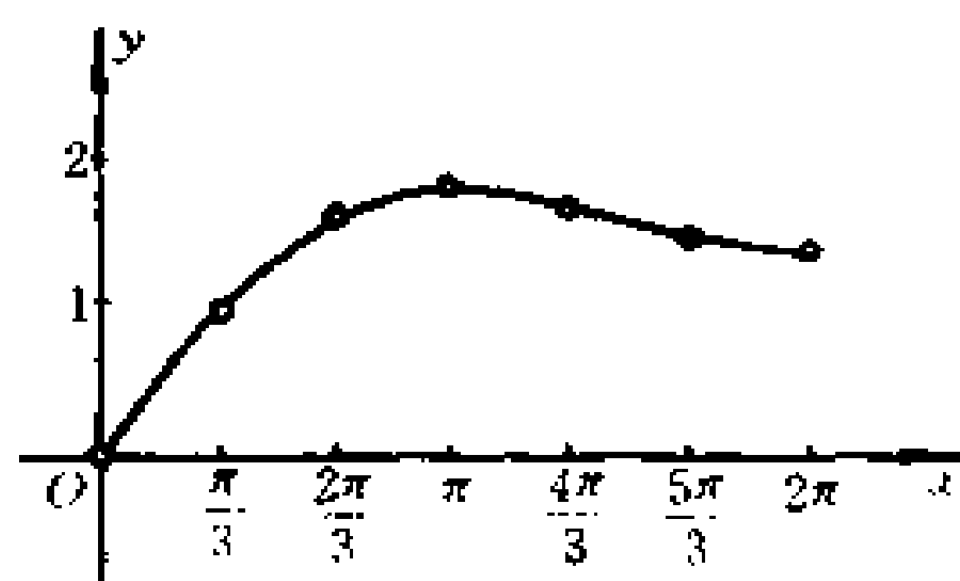


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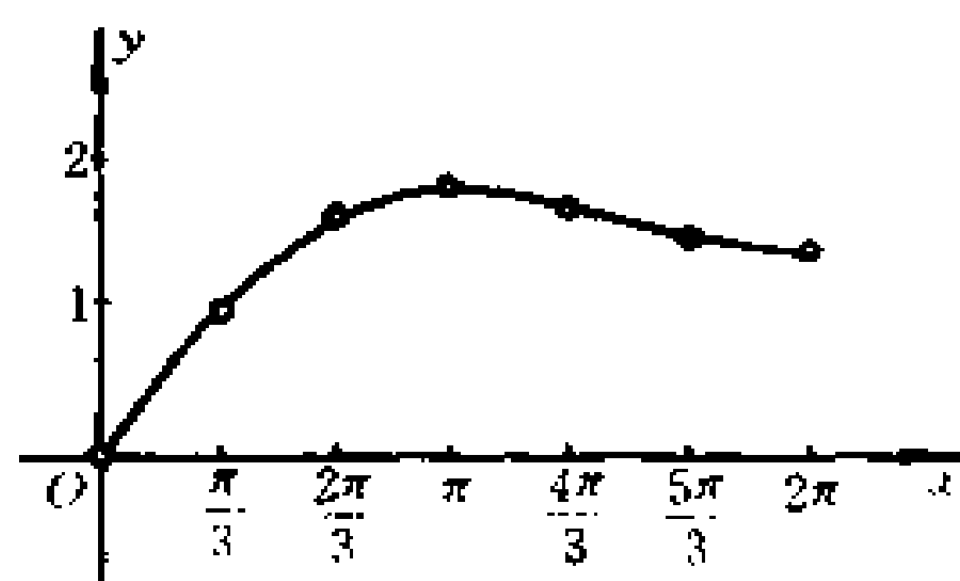


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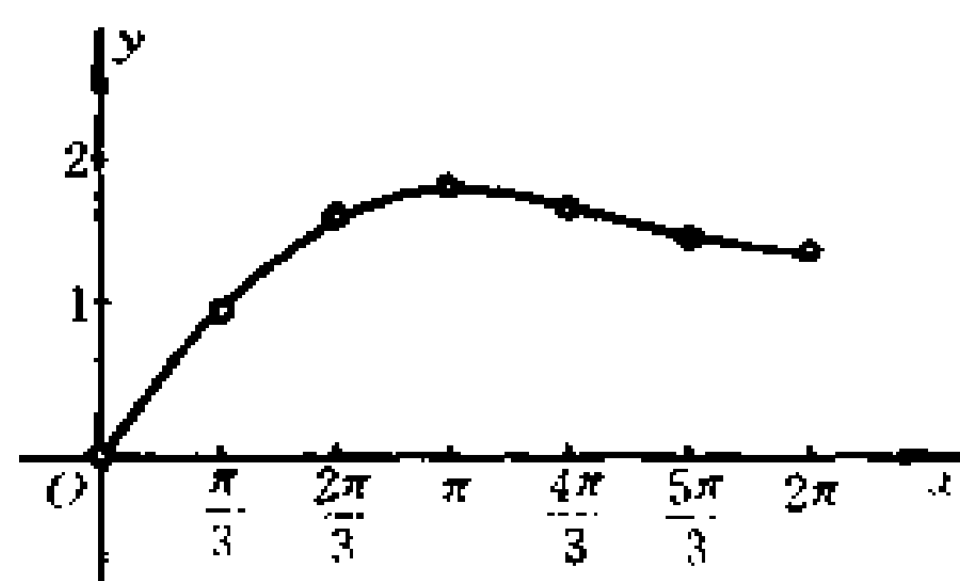


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$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

列表作图如下(图 4.55):

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

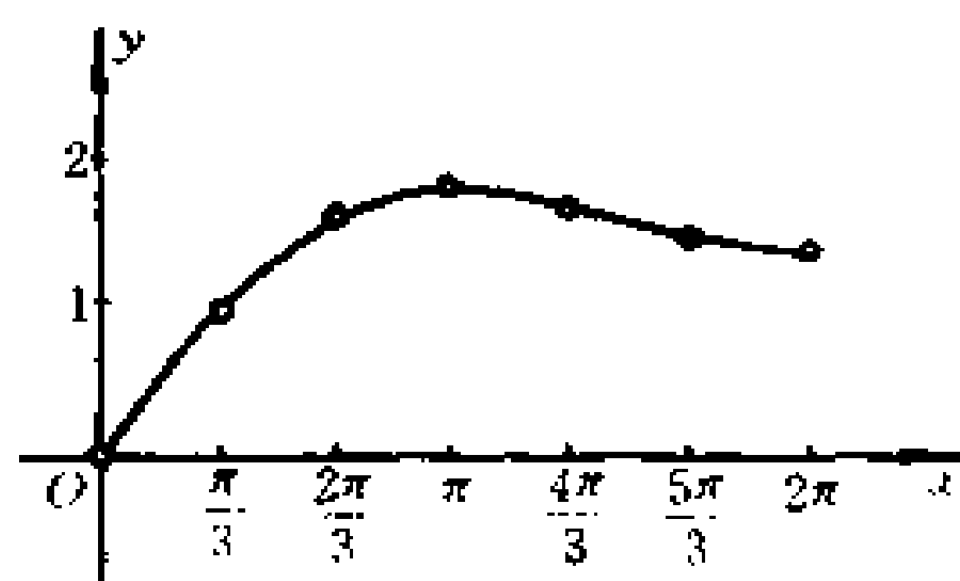


图 4.55

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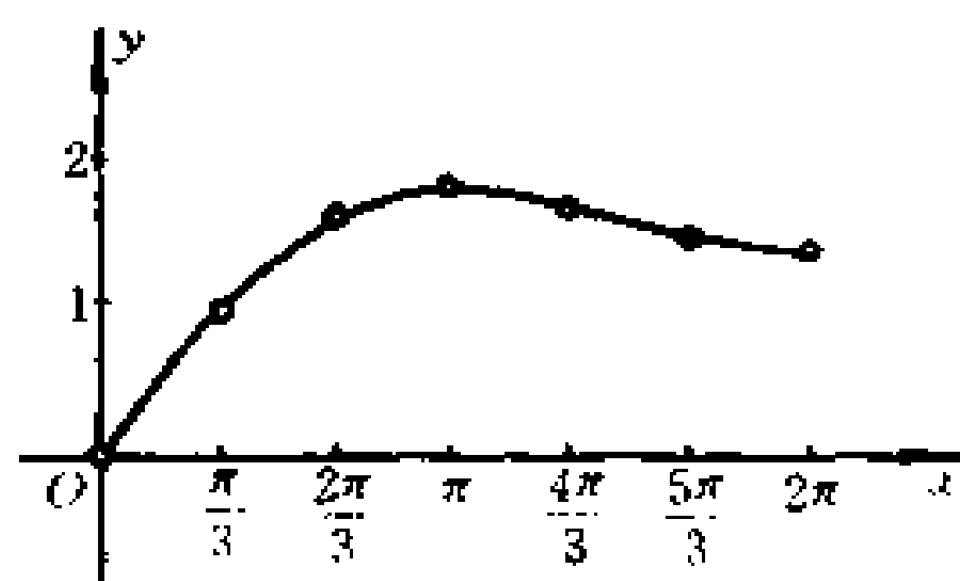


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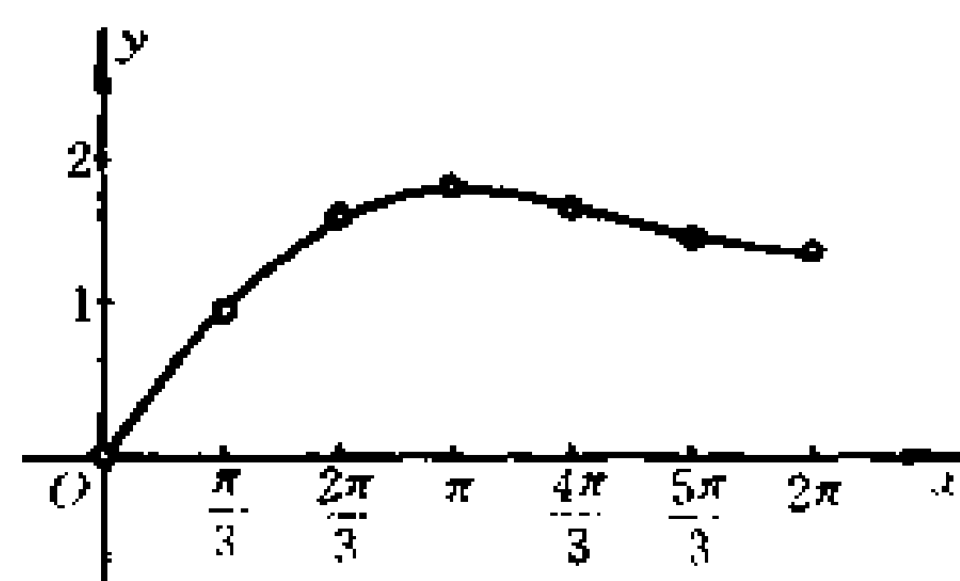


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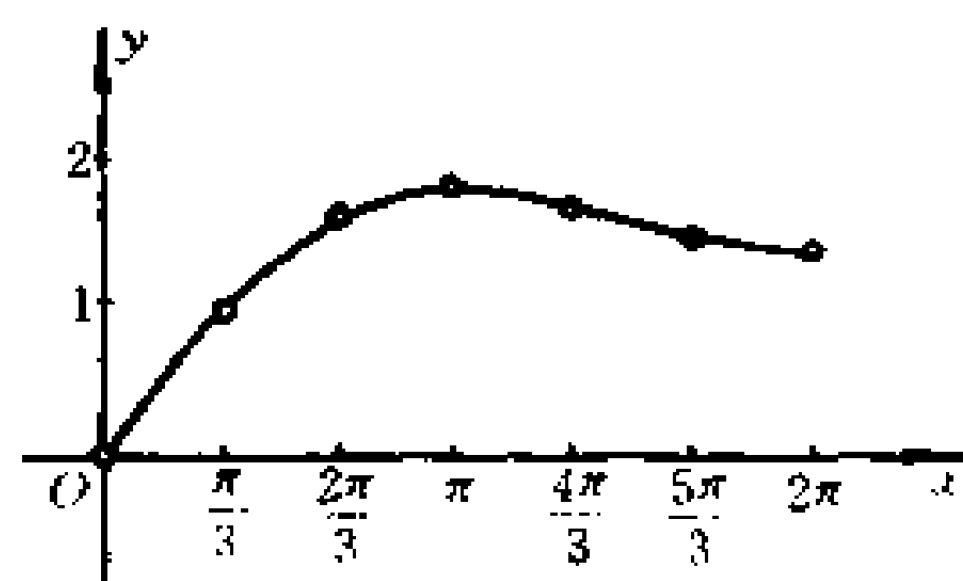


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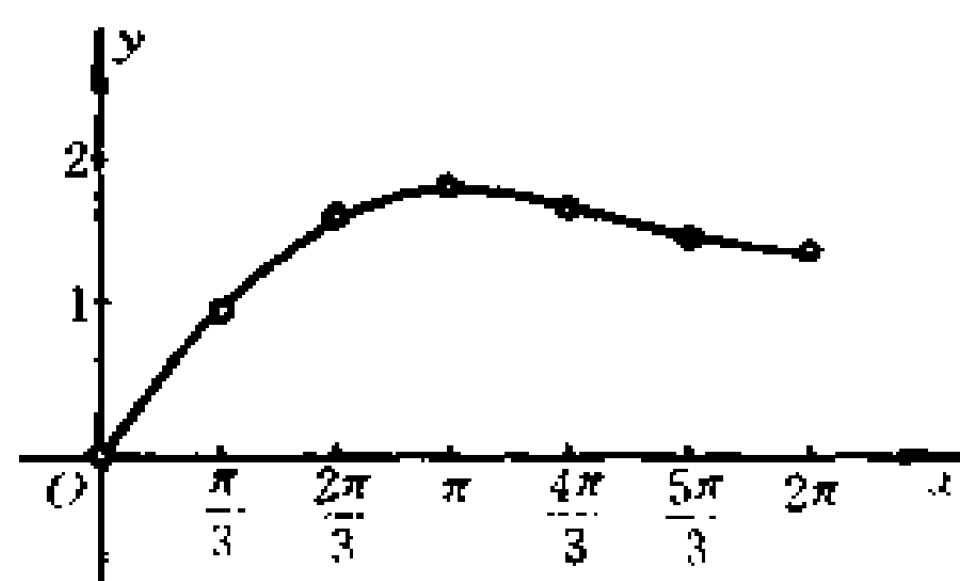


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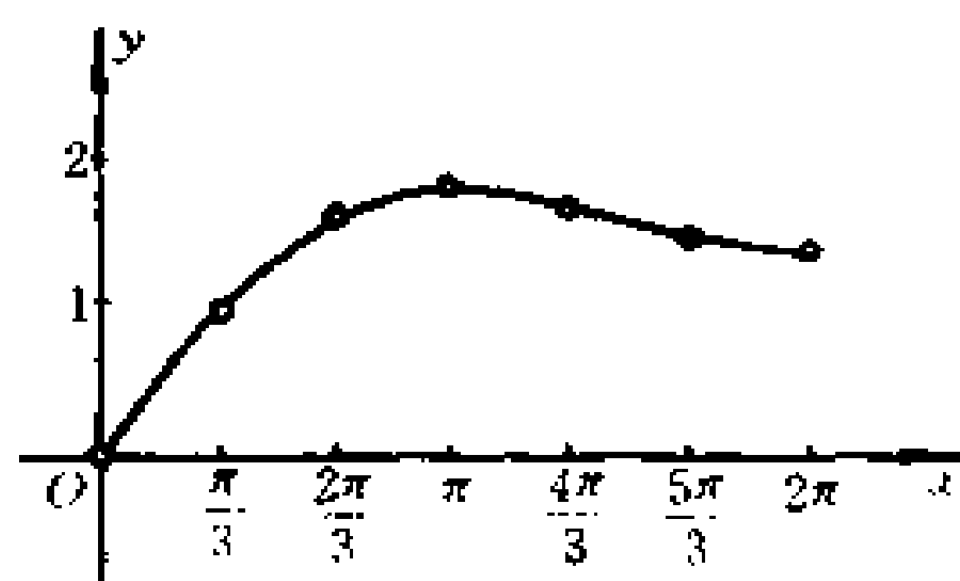


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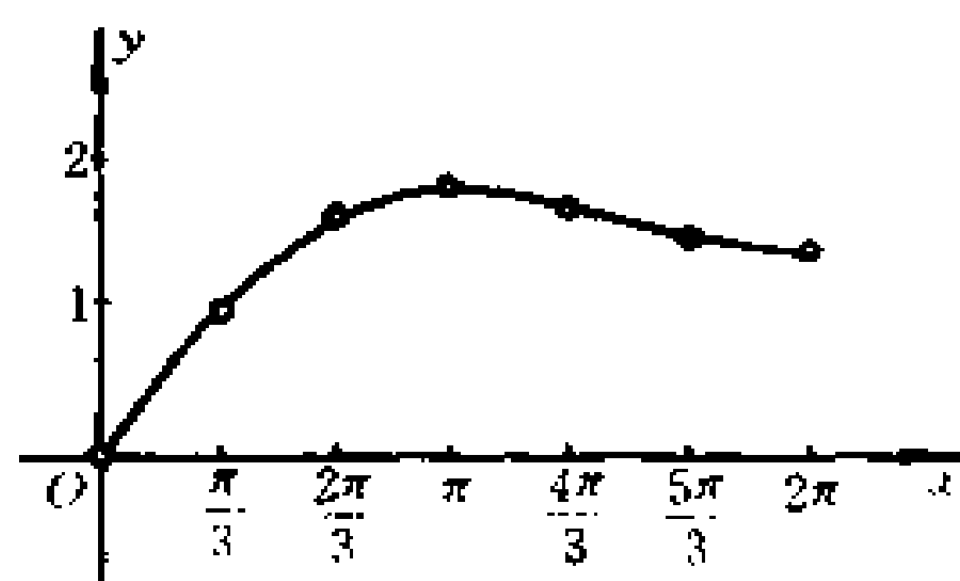


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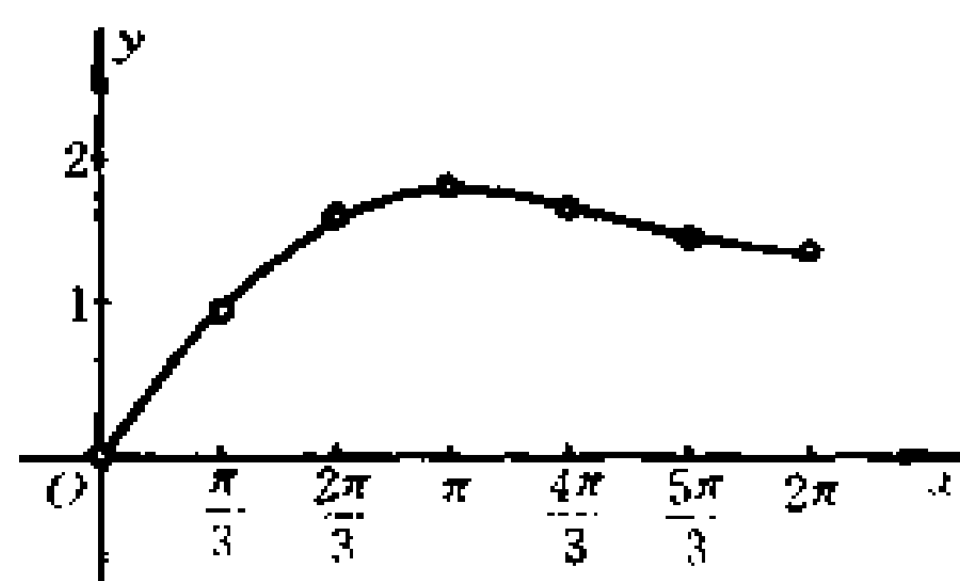


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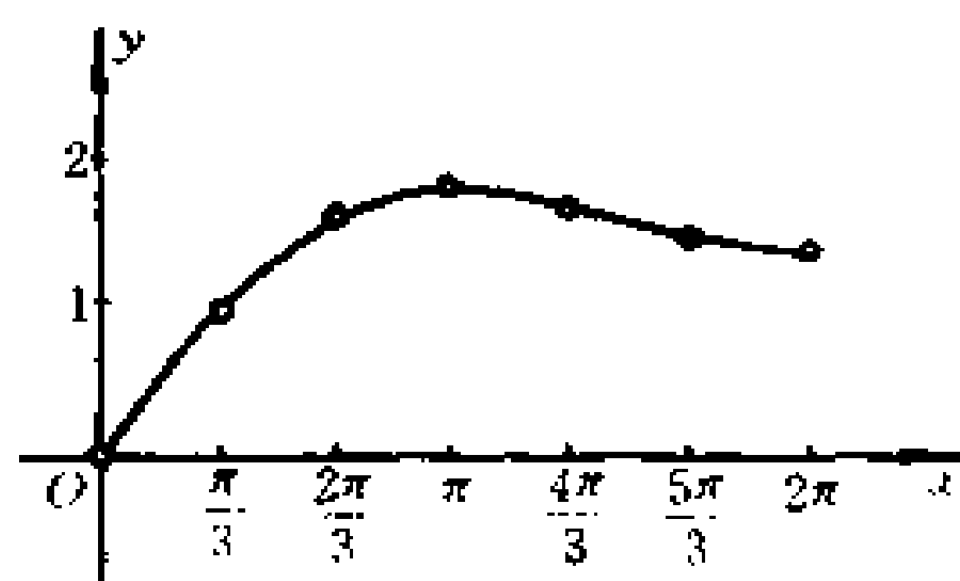


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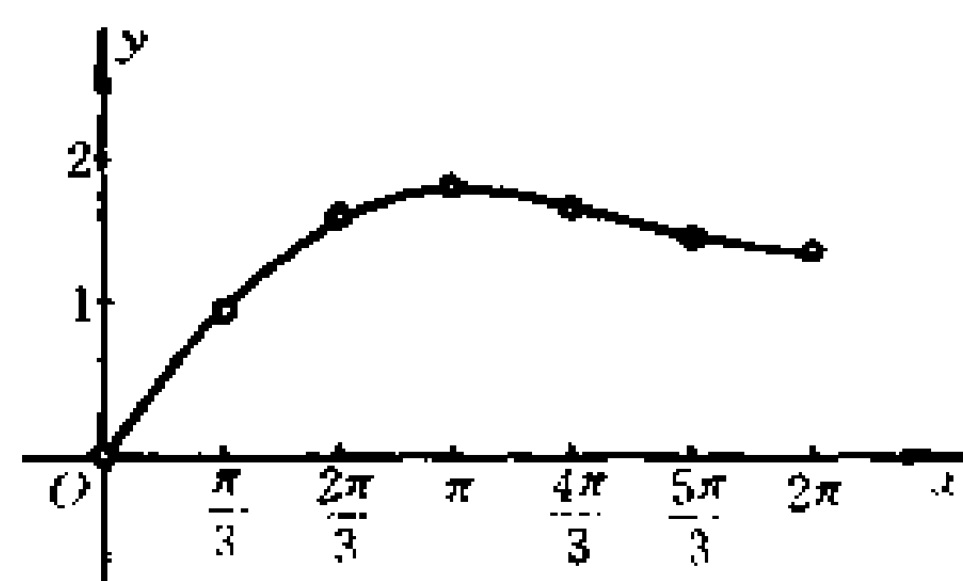


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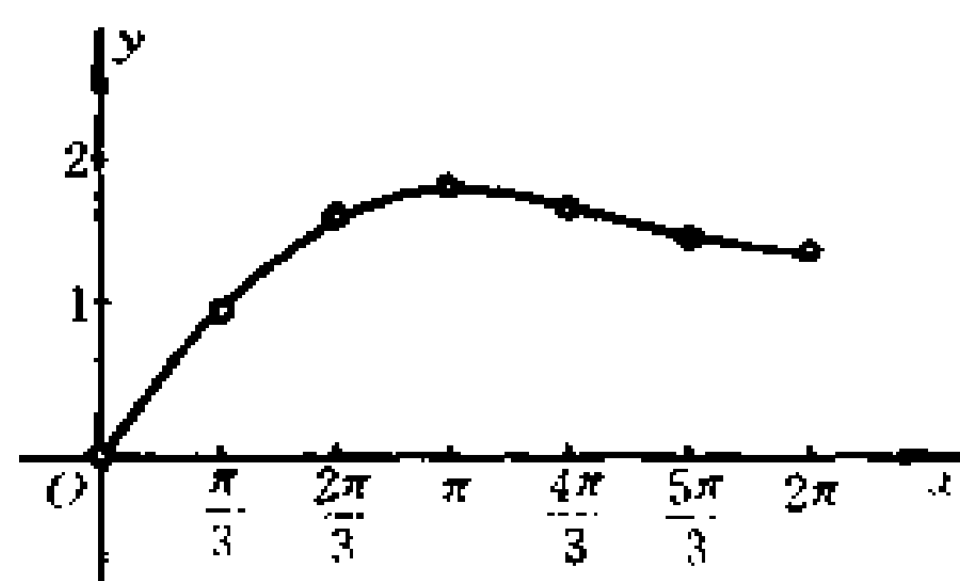


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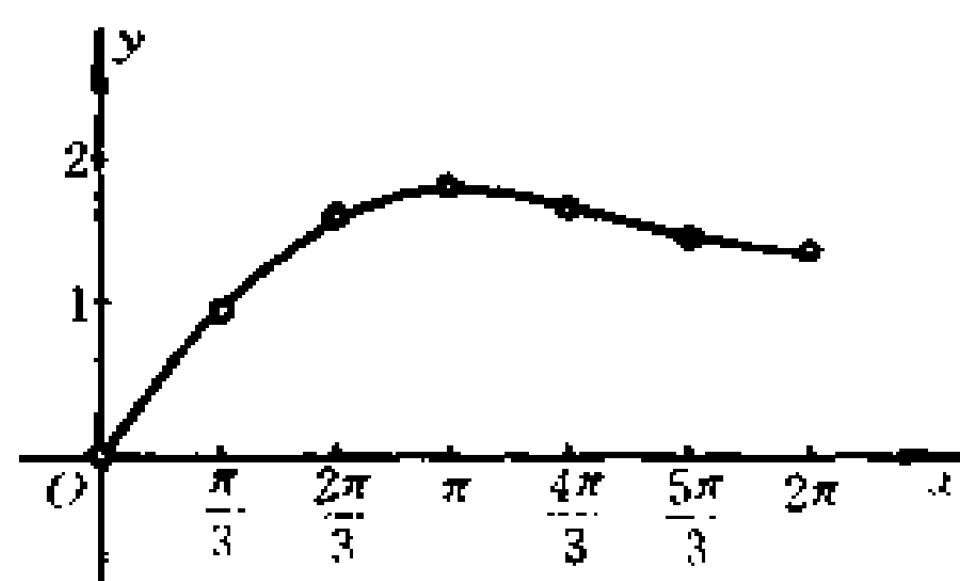


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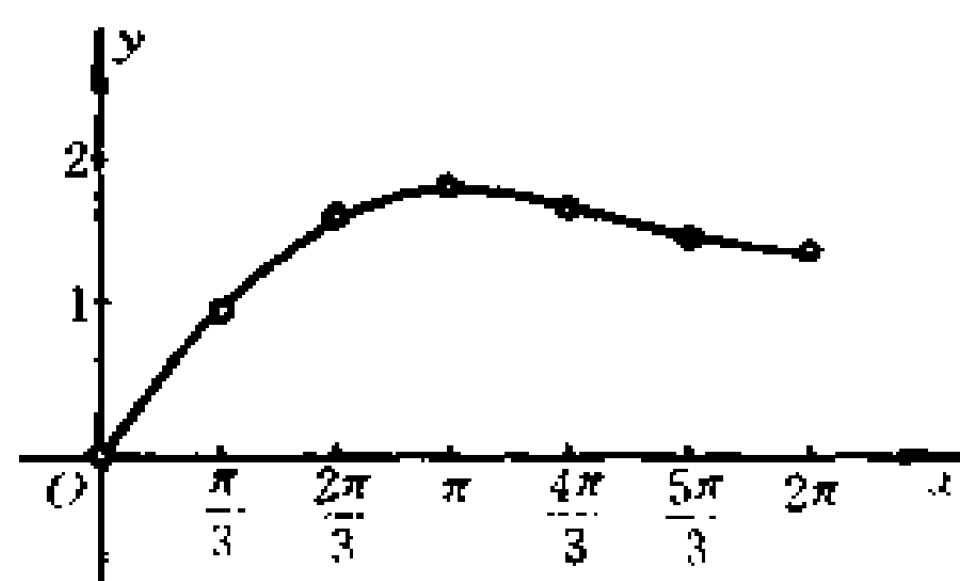


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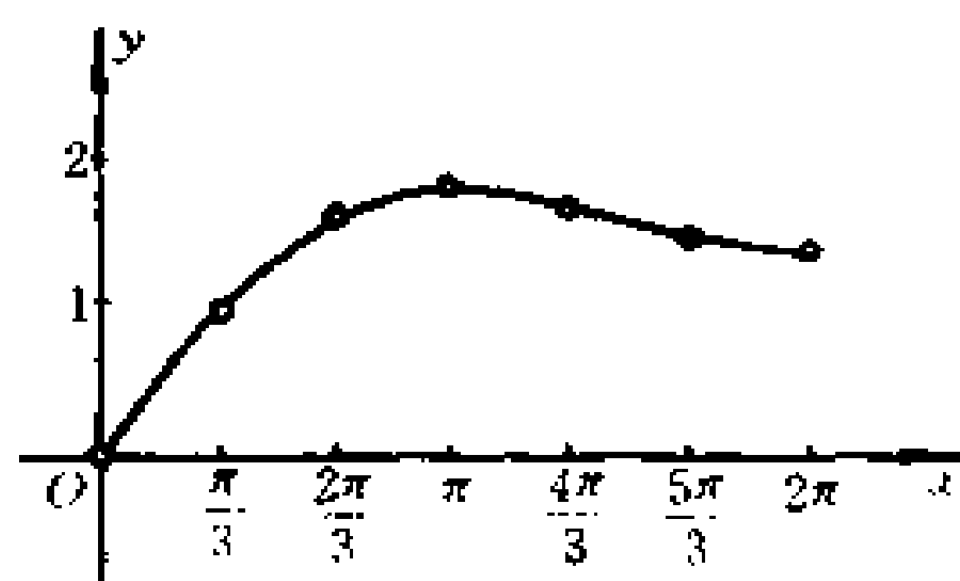


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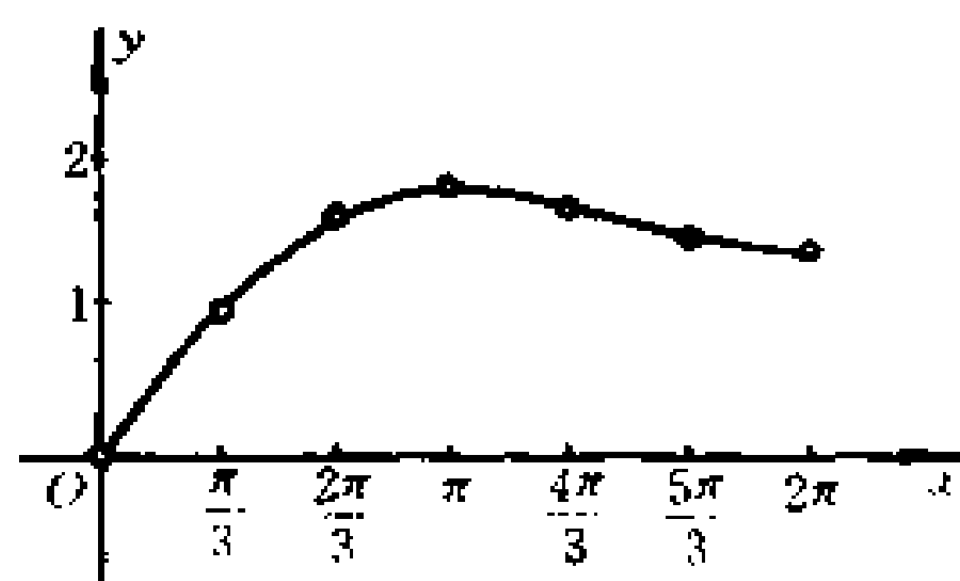


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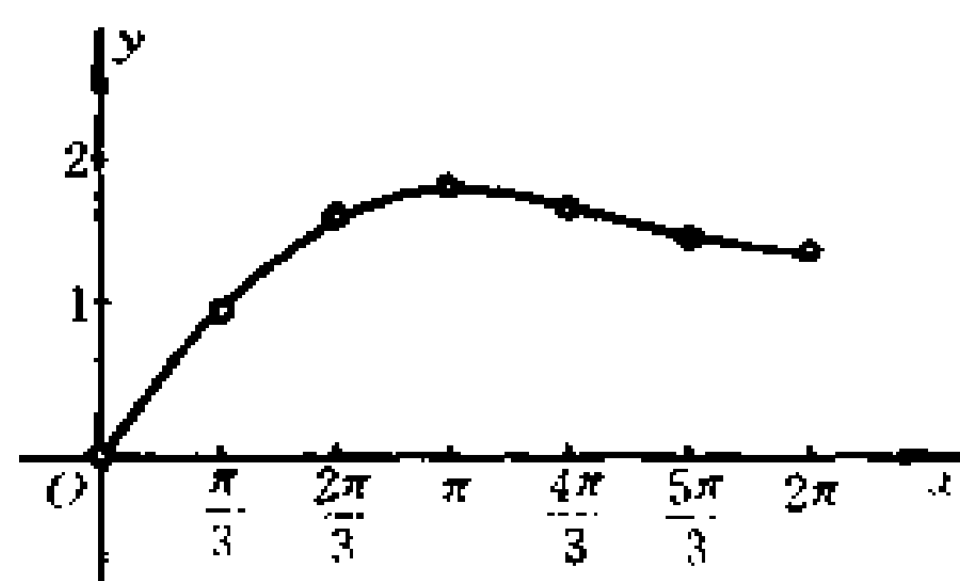


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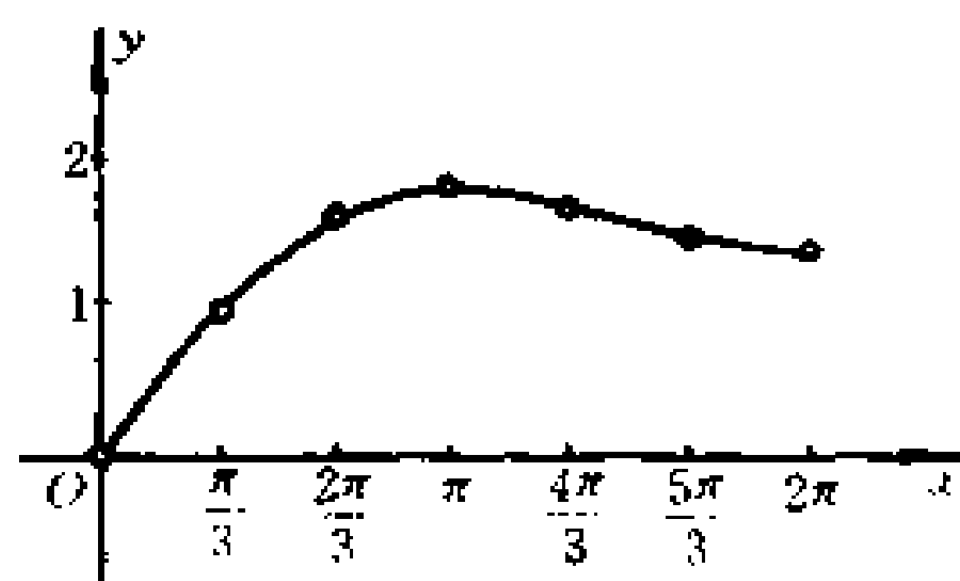


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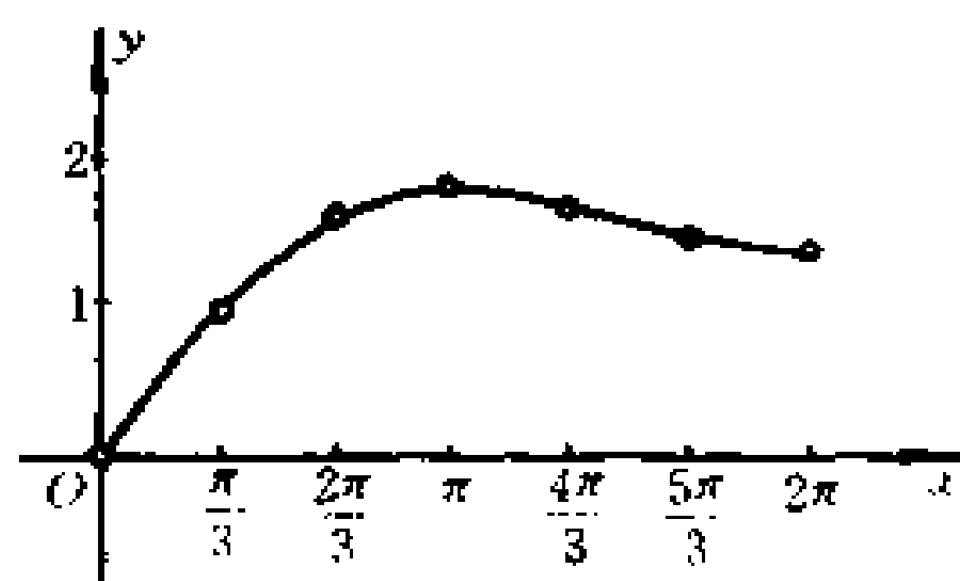


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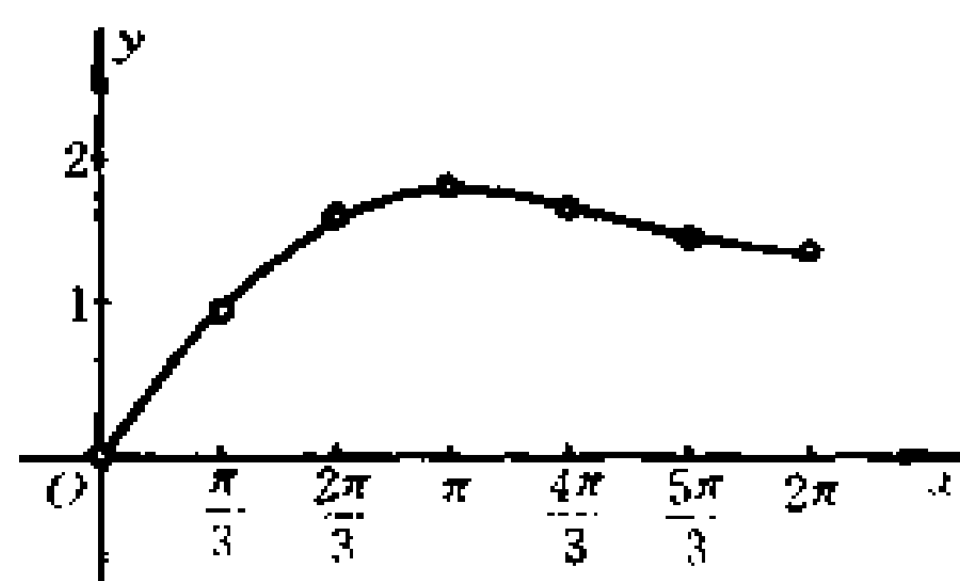


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$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

列表作图如下(图 4.55):

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

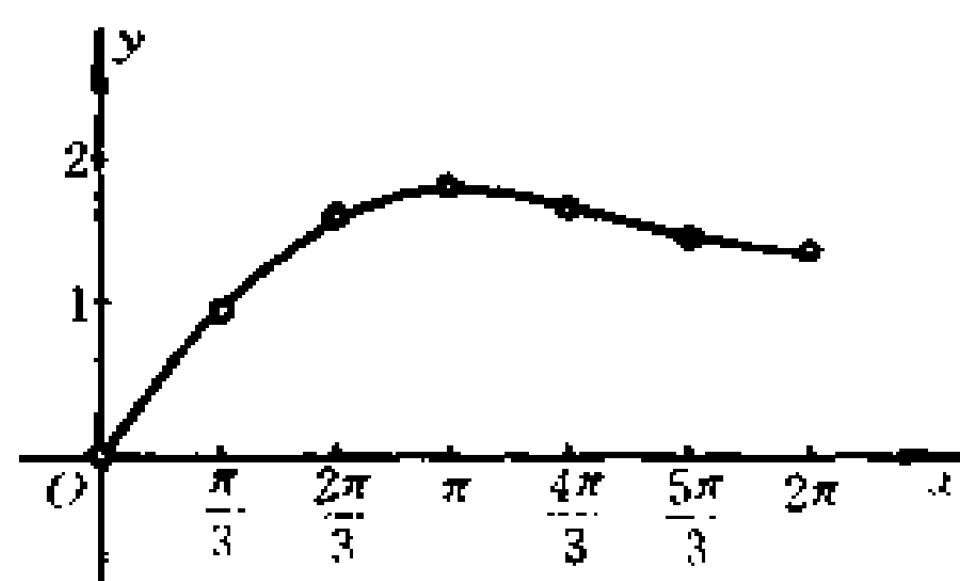


图 4.55

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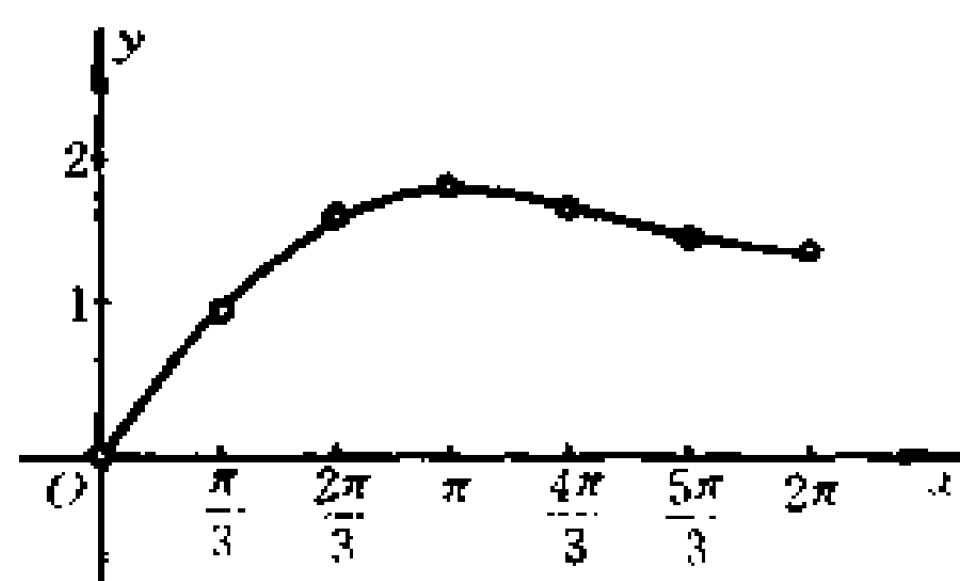


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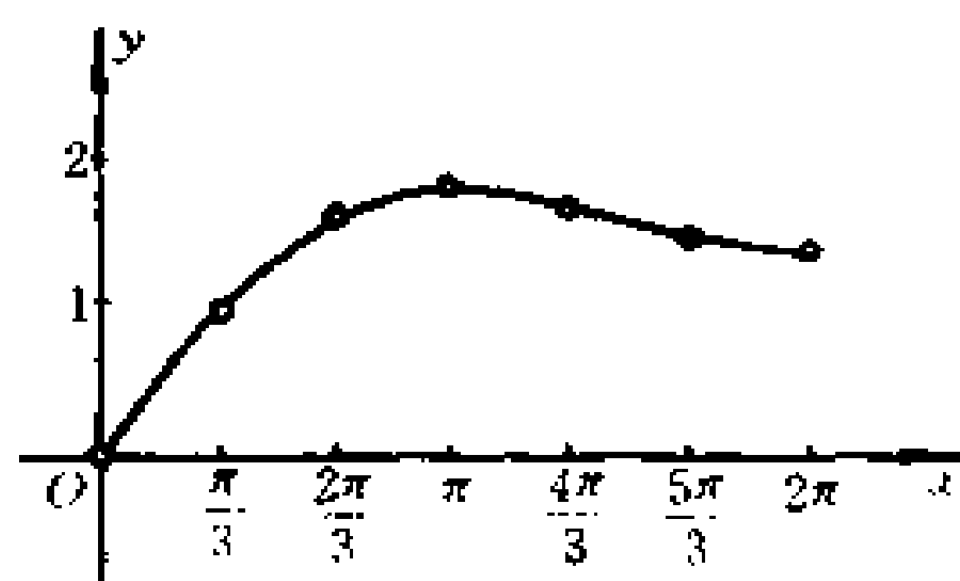


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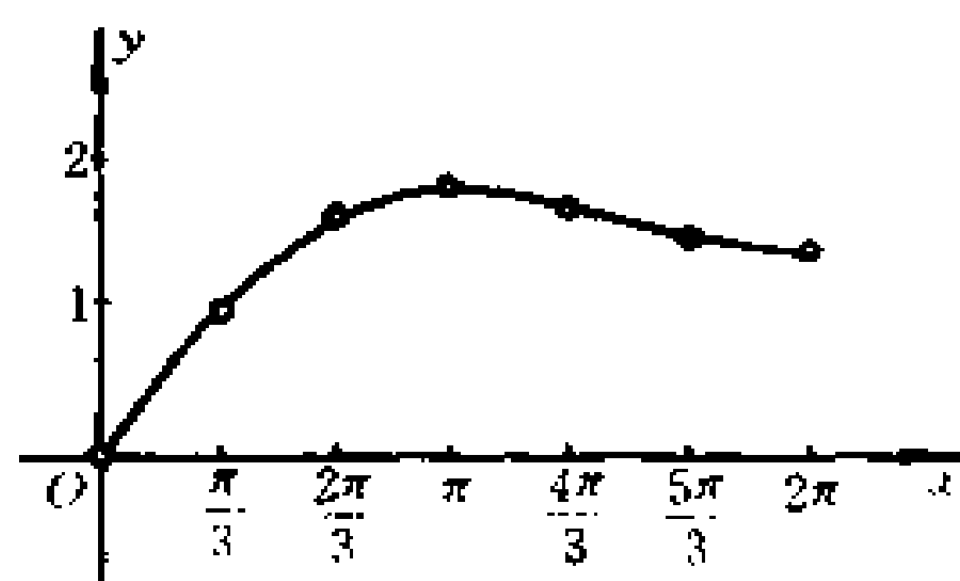


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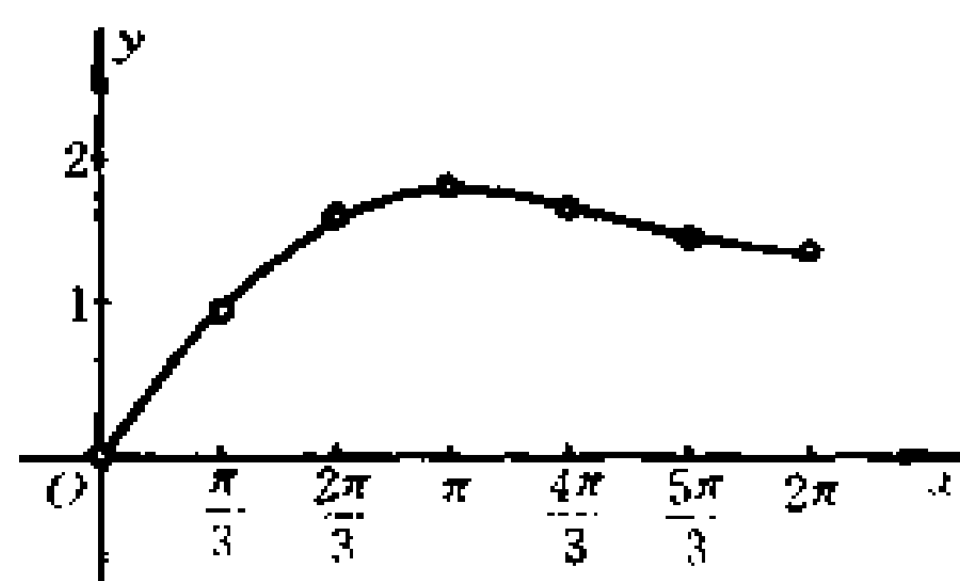


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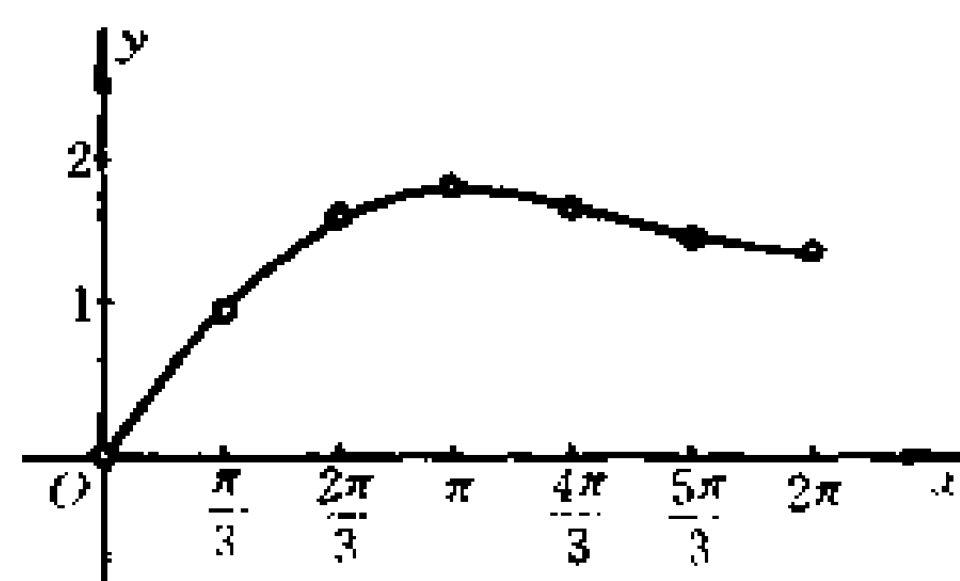


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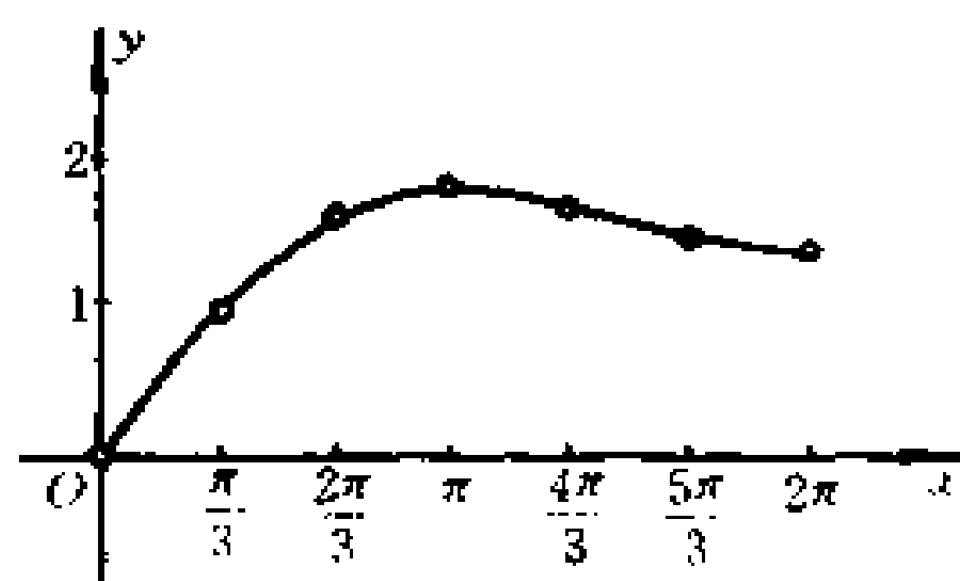


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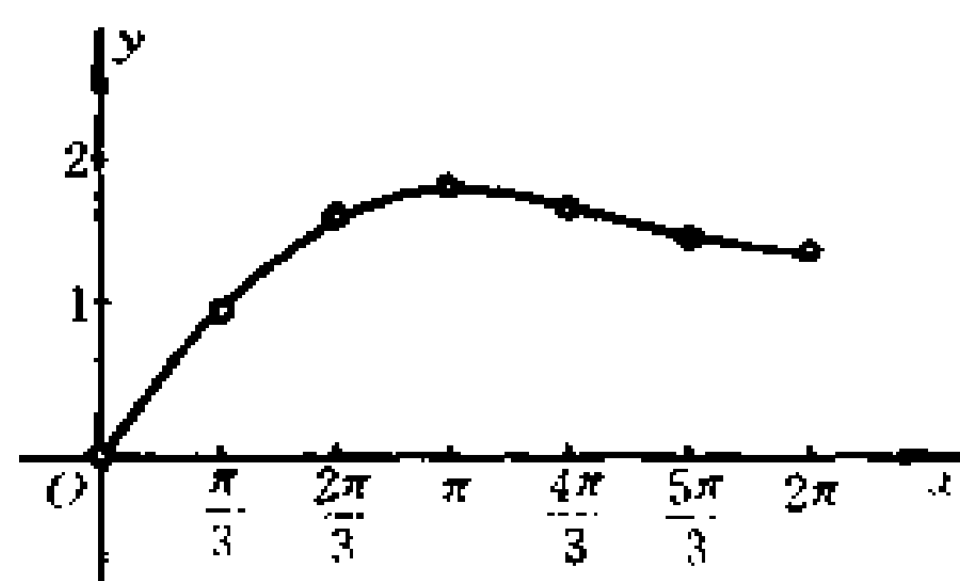


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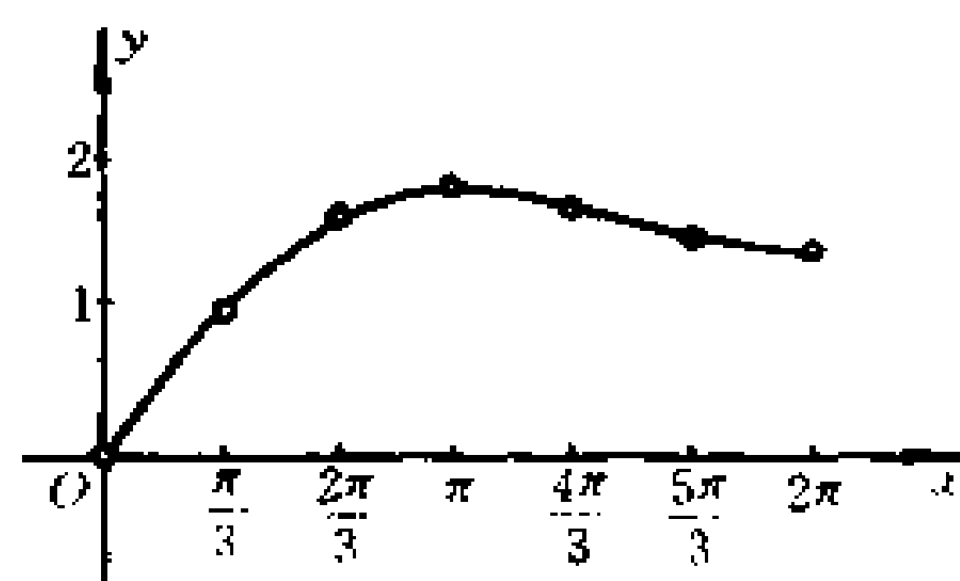


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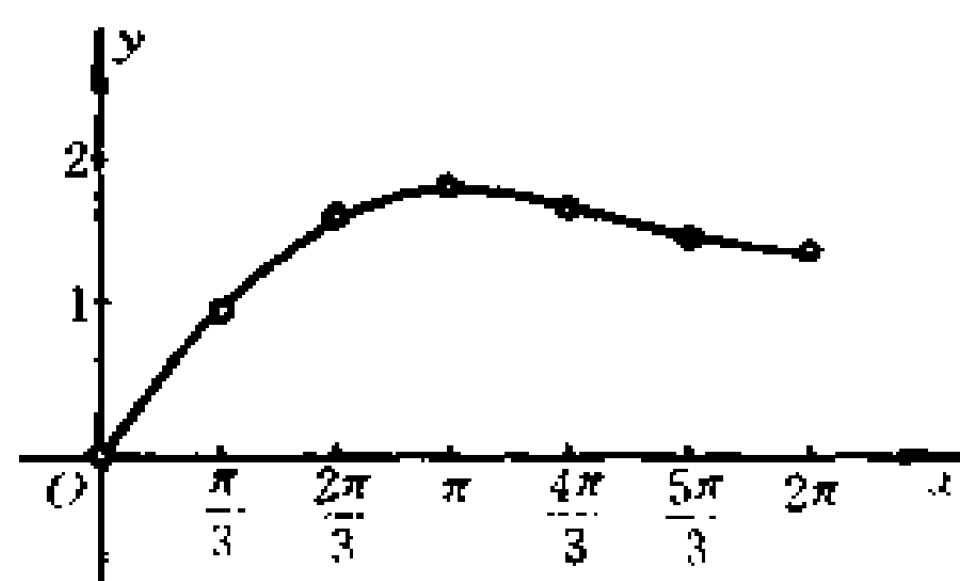


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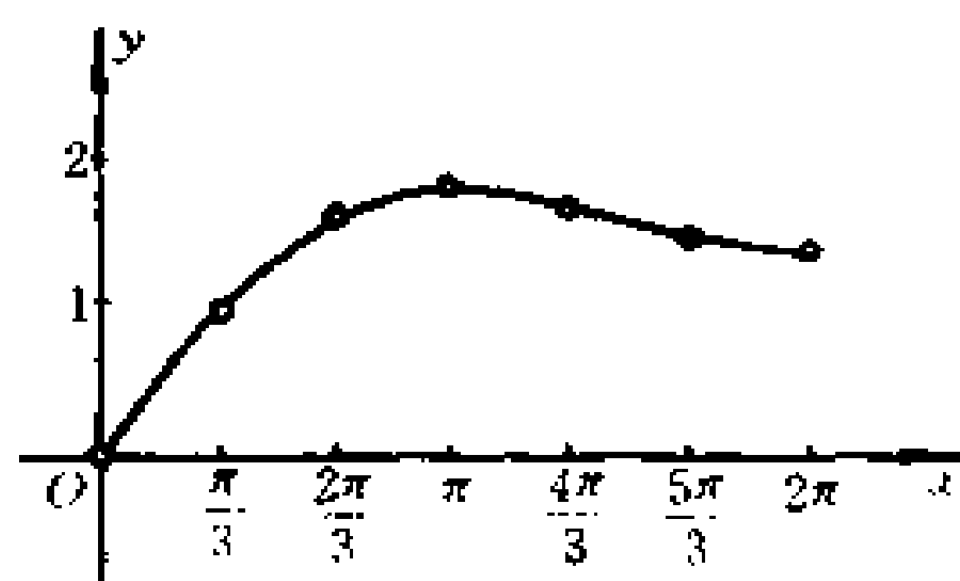


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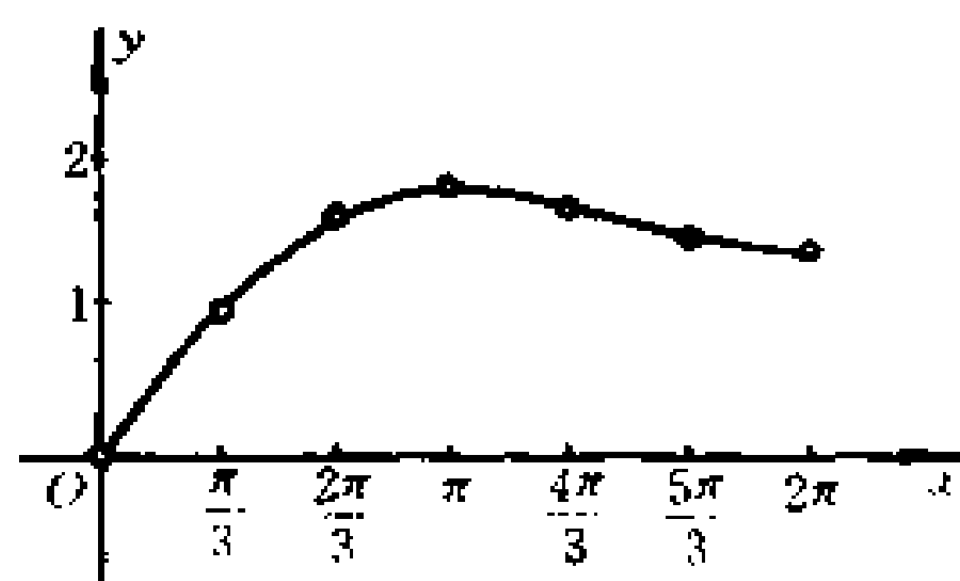


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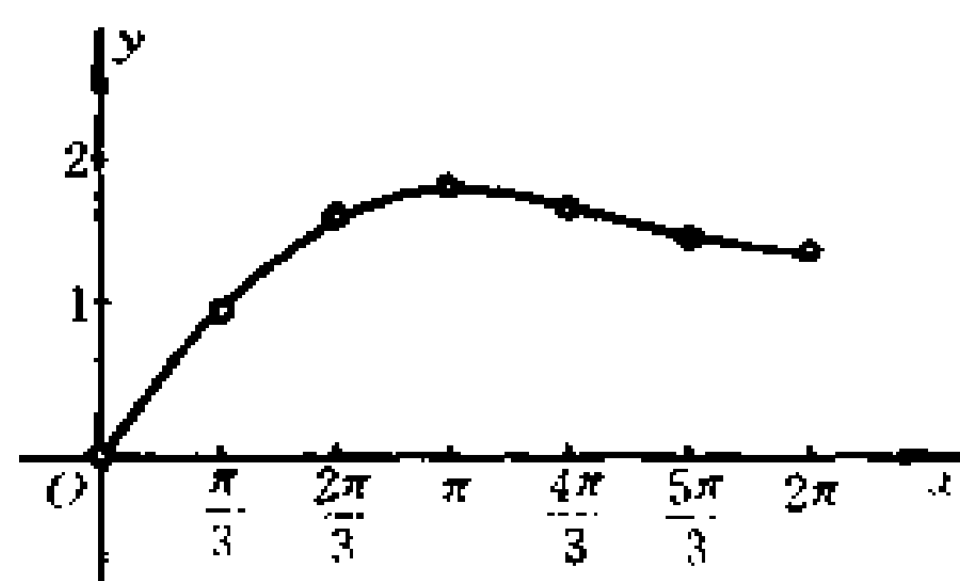


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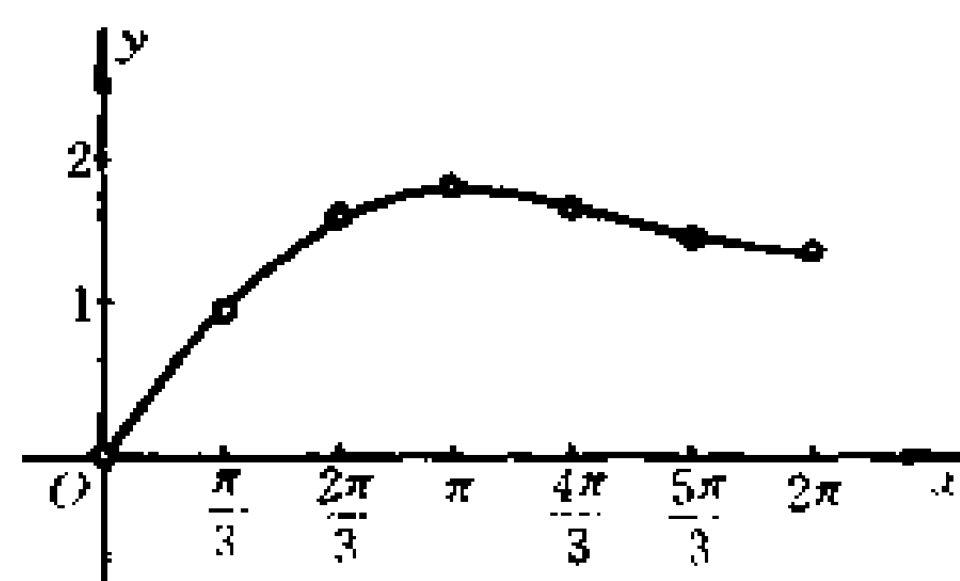


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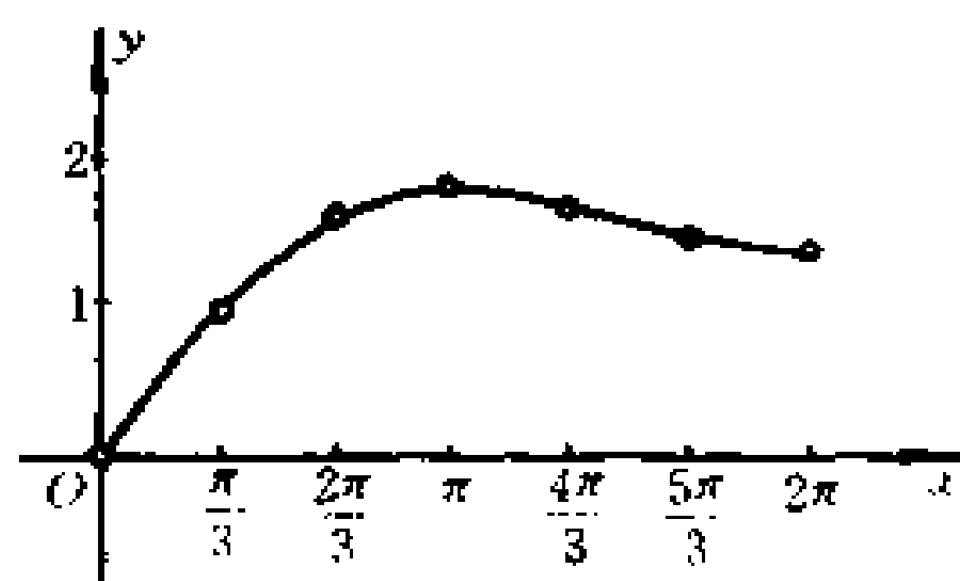


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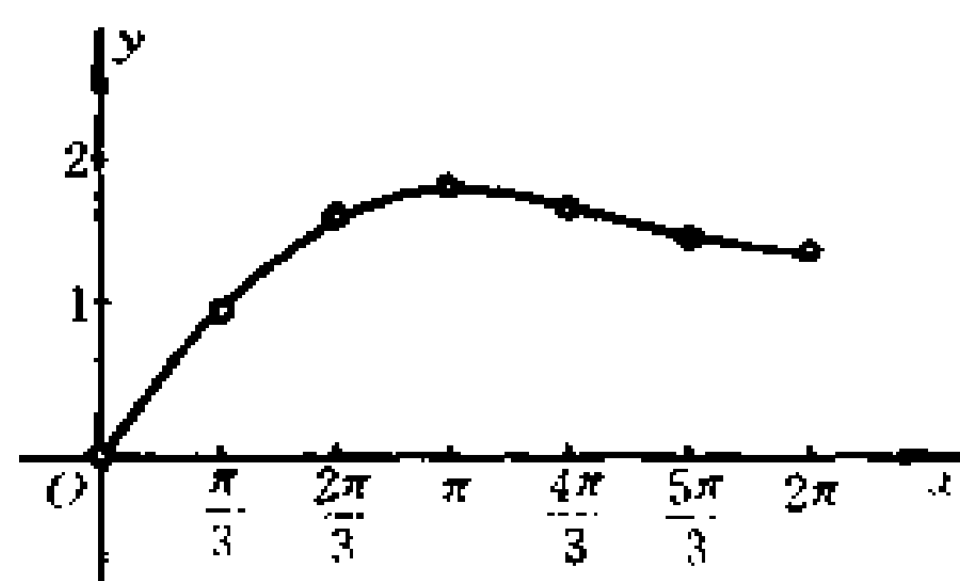


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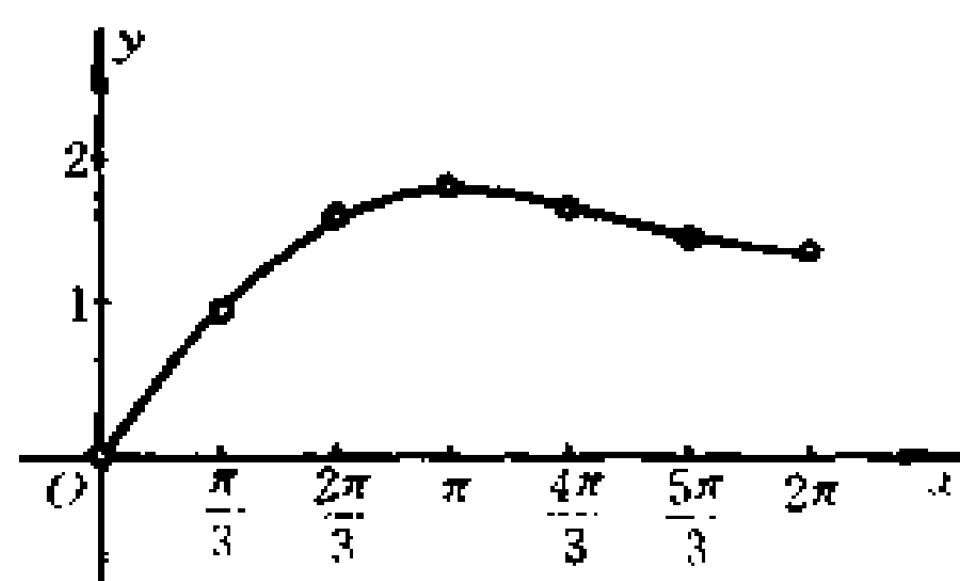


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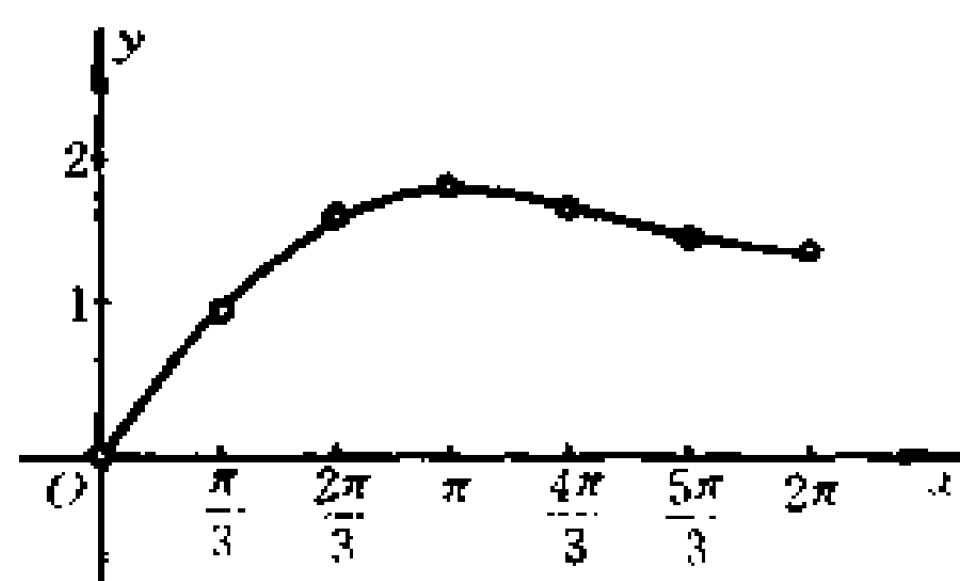


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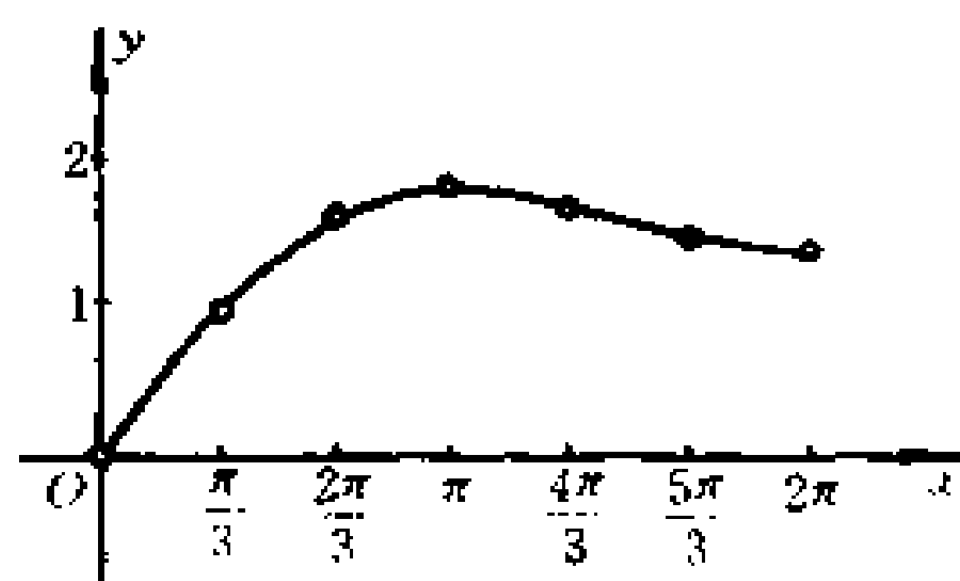


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$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

列表作图如下(图 4.55):

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

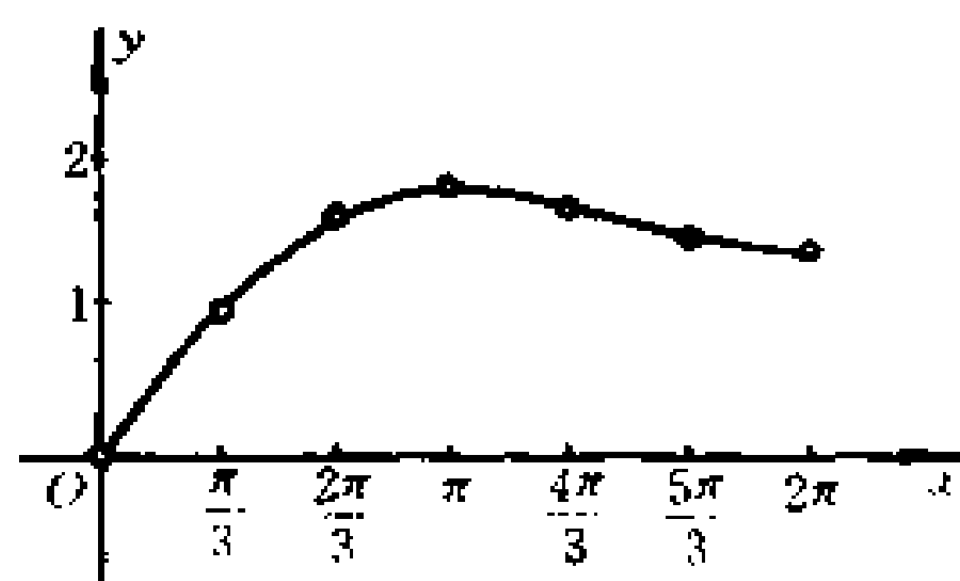


图 4.55

$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{27} (1 + 0.413 + 3.919 + 3.308 \\ &\quad + 2.257 + 1.841 + 1.411) \\ &\doteq 1.65.\end{aligned}$$

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|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

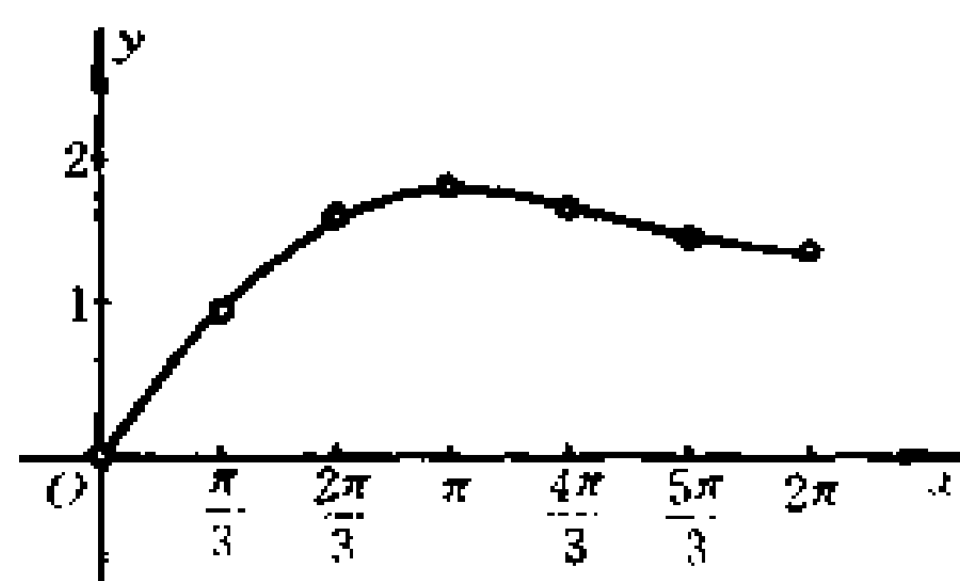


图 4.55

$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{27} (1 + 0.413 + 3.919 + 3.308 \\ &\quad + 2.257 + 1.841 + 1.411) \\ &\doteq 1.65.\end{aligned}$$

选取适当的 n , 类似地可求得

$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

列表作图如下(图 4.55):

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

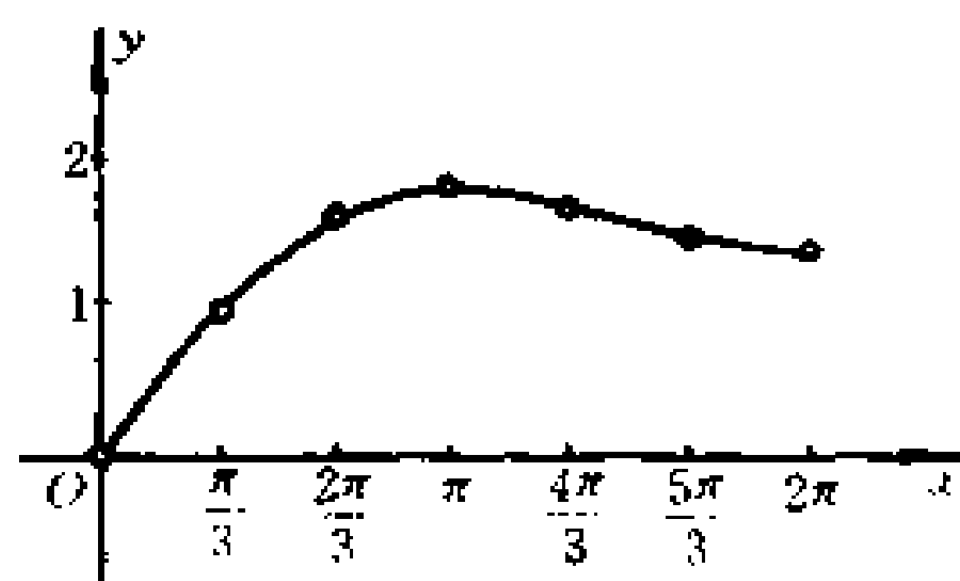


图 4.55

$$\begin{aligned}\int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{27} (1 + 0.413 + 3.919 + 3.308 \\ &\quad + 2.257 + 1.841 + 1.411) \\ &\doteq 1.65.\end{aligned}$$

选取适当的 n , 类似地可求得

$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{t} dt &\doteq 1.85, & \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.72 \\ \int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt &\doteq 1.52; & \int_0^{2\pi} \frac{\sin t}{t} dt &\doteq 1.42.\end{aligned}$$

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| y | 0 | 0.99 | 1.65 | 1.85 | 1.72 | 1.52 | 1.42 |

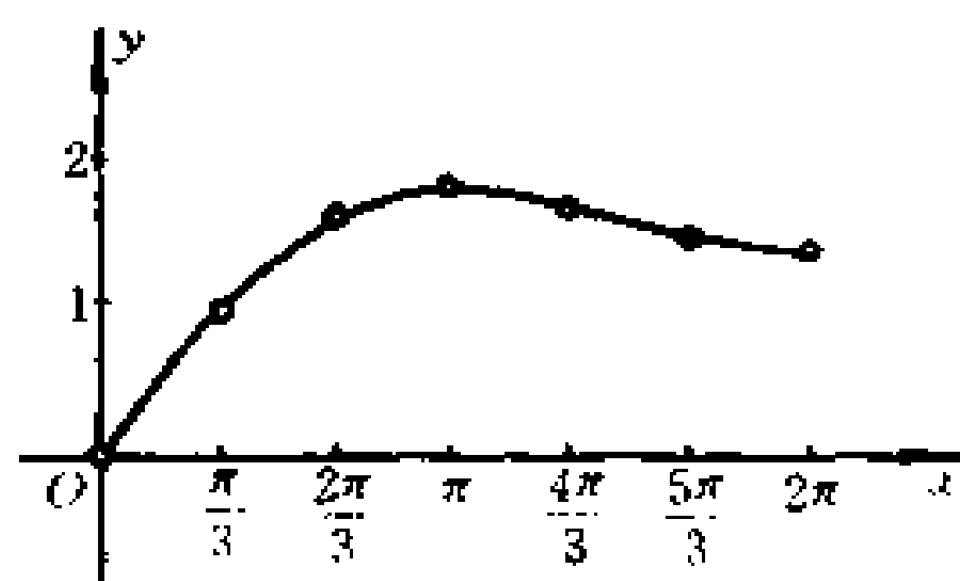


图 4.55